Al 1.2
LADS 6
6

Row Reduction

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}}_{A} \xrightarrow[-3R1+R_4 \to R_4]{-2R1+R_2 \to R_2,\ -4R_1+R_3 \to R_3} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}$$

$$\xrightarrow[-4R2+R_4 \to R_4]{-3R2+R_3 \to R_3} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{-R3+R4 \to R4} \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{U}$$

We will see that this implies

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 3 & 1 & 0 \\ 3 & 4 & 1 & 1 \end{bmatrix}}_{L = \text{lower}} \underbrace{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}}_{U = \text{upper}}$$

LADS 6

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Why does the procedure given yield the LU-decomposition?

We illustrate with the example

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \xrightarrow[-3R1+R_4 \to R_4]{\substack{-2R1+R2 \to R2 \\ -4R1+R3 \to R3}} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} 2$$

$$\underline{\underline{\text{OR}}} \quad \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot[2\ 1\ 1\ 0] \\ 2\cdot[2\ 1\ 1\ 0] \\ 4\cdot[2\ 1\ 1\ 0] \\ 3\,[2\ 1\ 1\ 0] \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 0 \end{bmatrix}}_{\text{outer product}} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix} \xrightarrow[-4R2+R4 \to R4]{-3R2+R3 \to R3}$$

$$= \begin{bmatrix}1\\2\\4\\3\end{bmatrix}\begin{bmatrix}2&1&1&0\end{bmatrix} + \begin{bmatrix}0\\1\\3\\4\end{bmatrix}\begin{bmatrix}0&1&1&1\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&2&2\\0&0&2&4\end{bmatrix} \xrightarrow{\substack{-2R3+R4\\ \to R4}}$$

(3)

$$= \begin{bmatrix}1\\2\\4\\3\end{bmatrix}\begin{bmatrix}2&1&1&0\end{bmatrix} + \begin{bmatrix}0\\1\\3\\4\end{bmatrix}\begin{bmatrix}0&1&1&1\end{bmatrix} + \begin{bmatrix}0\\0\\1\\2\end{bmatrix}\begin{bmatrix}0&0&2&2\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&2\end{bmatrix}$$

$$= \begin{bmatrix}1\\2\\4\\3\end{bmatrix}\begin{bmatrix}2&1&1&0\end{bmatrix} + \begin{bmatrix}0\\1\\3\\4\end{bmatrix}\begin{bmatrix}0&1&1&1\end{bmatrix} + \begin{bmatrix}0\\0\\1\\2\end{bmatrix}\begin{bmatrix}0&0&2&2\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\begin{bmatrix}0&0&0&2\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&0\\2&1&0&0\\4&3&1&0\\3&4&2&1\end{bmatrix}\begin{bmatrix}2&1&1&0\\0&1&1&1\\0&0&2&2\\0&0&0&2\end{bmatrix} = L\,U$$

Special Cases –

(1) ~~not~~ $A = LU$ not unique

$= (\frac{1}{2}L)(2U)$ still in form

Lower · Upper.

(2) Sometimes row exchanges are needed for

(a) deal with zeros

(b) deal with numerical efficiency

Pivoting

example

$$A = \begin{bmatrix} \boxed{0} & 3 & 2 \\ 1 & 2 & 1 \\ -1 & 0 & 3 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ permutation matrix

so general form is $PA = LU$

$\Rightarrow A = P^{-1}LU$, $\det(A) = \det(P^{-1})(\det L)(\det U)$

(3) Algorithm yields a row of zeroes

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Algorithm terminates $\Leftrightarrow$ happens when A is non-invertible $\Leftrightarrow$ $\operatorname{rank}(A) < n$

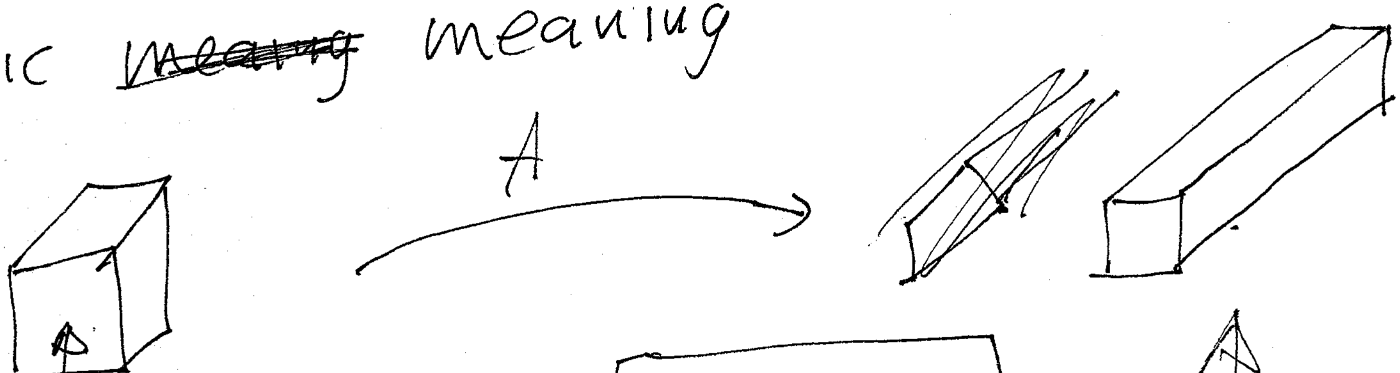
LU-Decomposition

Two more applications of $A = LU$

1. Computing $\det(A)$, determinant

What is $\det(A)$ - Formula — ....

geometric ~~meaning~~ meaning



R = Region in $\mathbb{R}^n$

A is $n \times n$

$\mathrm{vol}(R)$

vol(A(R)) = $\underline{\det(A)}$ $\mathrm{vol}(R)$

properties of det

(2) $\det(AB) = \det(A)\det(B)$

$$\begin{tikzcd} \cdot \arrow[r, "B"] \arrow[rr, bend right, "AB"'] & \cdot \arrow[r, "A"] & \cdot \end{tikzcd}$$

(3) If T is triangular (upper or lower)
$\Rightarrow \det(T) = T_{11} T_{22} \cdots T_{nn}$ = product of diagonal product.

Putting these together

$$\det(A) = \det(L\,U)$$

$$= \det(L) \cdot \det(U)$$

$$= L_{11} \cdots L_{nn} \cdot U_{11} \cdots U_{nn}.$$

For the example

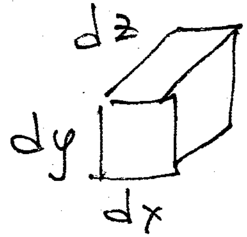
$$\det(A) = 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 2 = \underline{8}$$

The Inverse – Find $\underline{\underline{X}}$ so that

$$A\underline{\underline{X}} = \underline{\underline{I}} \rightarrow \text{Treat this as } n\text{-equations}$$

$$A\begin{bmatrix} \vec{X}_1 & \vec{X}_2 & \cdots & \vec{X}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix}$$

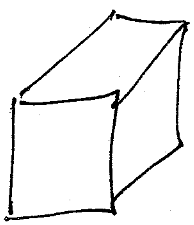
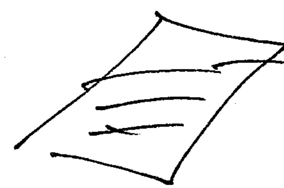
$$\Rightarrow \left.\begin{aligned} A\vec{x}_1 &= \vec{e}_1 \\ A\vec{x}_2 &= \vec{e}_2 \\ &\vdots \\ A\vec{x}_n &= \vec{e}_n \end{aligned}\right\} \text{Solve these toget } \underline{\underline{X}} = A^{-1} \text{ using LU decomp.}$$



$$\xrightarrow{f} \underbrace{\left|\frac{\partial f_i}{\partial x_j}\right|}_{\det \,=\, \text{Jacobian.}} dx\,dy\,dz$$

der of f.

eq $\det = 0$

 $\xrightarrow{A}$  $A(R)$

no vol.

<u>no inverse</u>