Orthogonality cont.

$\vec{u}\cdot\vec{v}=0 \qquad \vec{u}\perp\vec{v} \qquad \vec{u}^T\vec{v} \qquad \langle\vec{u},\vec{v}\rangle$

$\{\vec{v}_1,\dots,\vec{v}_k\}$ is called <u>orthogonal</u> if

$$\vec{v}_i\circ\vec{v}_j=0 \quad \text{when } i\neq j$$

$\{\vec{u}_1,\vec{u}_2,\dots,\vec{u}_k\}$ is orthonormal (o.n.)

if (1) it's orthogonal

(2) $\|\vec{u}_i\|=1$ for all i

OR

$$\vec{u}_i \cdot \vec{u}_j = \delta_{ij} \quad \text{(Kronecker delta)}$$

$$\delta_{ij} = 0 \quad \text{if } i \neq j$$
$$= 1 \quad \text{if } i = j$$

$$\vec{u}_i \cdot \vec{u}_i = \|\vec{u}_i\|^2 = 1$$

~~even~~

Coordinates are ~~clear~~ to compute in a orthonormal basis

$$\vec{W} = \sum \alpha_i \vec{u}_i$$

easy

$$\vec{w} = \sum \alpha_i \vec{u}_i$$

α_i coordinates

$$\vec{u}_j \cdot \vec{w} = \sum \alpha_i \vec{u}_j \cdot \vec{u}_i$$

$$= \alpha_j \vec{u}_j \cdot \vec{u}_j = \alpha_j$$

$$\boxed{\alpha_j = \vec{w} \cdot \vec{u}_j}$$

formula for coordinate

eg: Fourier is the expansion ~~int~~ in the basis $\{\cos nt, \sin nt\}$

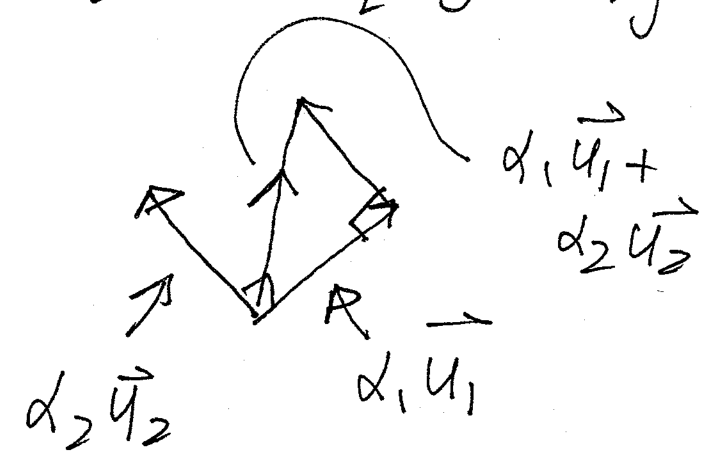
<u>Reminder</u>: $W \subseteq \mathbb{R}^n$ is a subspace

$\Rightarrow$ given a basis, Gram-Schmidt produces an o.n. one. $\to$ later QR decomp.

<u>Pythagorean Theorem</u>:

If $\vec{w} = \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2$ with $\vec{u}_i \cdot \vec{u}_j = \delta_{ij}$

$\Rightarrow \|\vec{w}\|^2 = \alpha_1^2 + \alpha_2^2$



Proof:

$$\|\vec{w}\|^2 = \vec{w}\cdot\vec{w} = (\alpha_1\vec{u}_1+\alpha_2\vec{u}_2)\cdot(\alpha_1\vec{u}_1+\alpha_2\vec{u}_2)$$

$$= \alpha_1^2\,\vec{u}_1\cdot\vec{u}_1 + \alpha_1\vec{u}_1\cdot\alpha_2\vec{u}_2 + \alpha_2\vec{u}_2\cdot\alpha_1\vec{u}_1$$

$$+ \alpha_2^2\,\vec{u}_2\cdot\vec{u}_2$$

$$= \alpha_1^2\cdot 1 + 0 + 0 + \alpha_2^2\cdot 1 = \alpha_1^2+\alpha_2^2$$

In general, if $\{\vec{u}_1, \dots, \vec{u}_k\}$, $\vec{u}_i\cdot\vec{u}_j = \delta_{ij}$

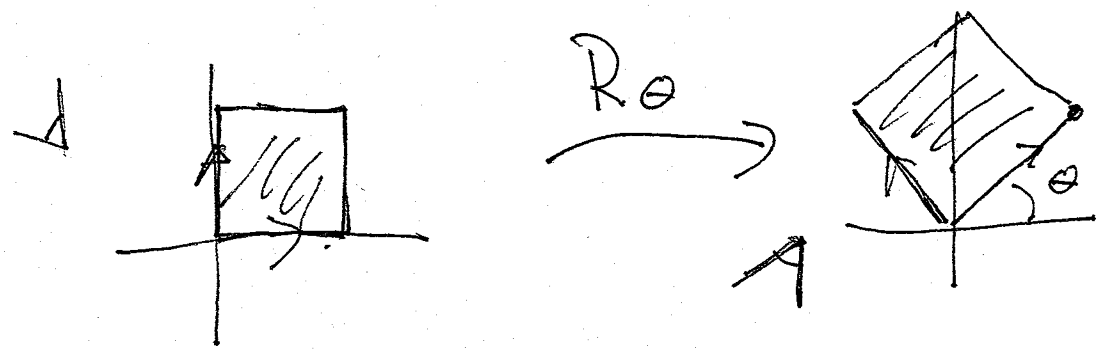
$$\vec{w} = \sum \alpha_i \vec{u}_i \Rightarrow$$

$$\|\vec{w}\|^2 = \sum \alpha_i^2$$

Orthogonal Matrix

A $n \times n$ matrix Q is orthogonal if its columns form an orthonormal basis for $\mathbb{R}^n$.

e.g. $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



Rigid rotation $\boxed{\text{counterclockwise}}$ by θ.

other equivalent definitions

(1) $Q^T Q = I$ OR $\boxed{Q^T = Q^{-1}}$

$$Q = \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}, \quad Q^T Q$$

$$= \begin{bmatrix} \vec{q}_1^{\,T} \\ \vec{q}_2^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix} = \begin{bmatrix} q_1^T q_1 & q_1^T \vec{q}_2 \\ & \end{bmatrix}$$

$$= \begin{bmatrix} \vec{q}_1 \cdot \vec{q}_1 & \vec{q}_1 \cdot \vec{q}_2 \\ & \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ & 1 & & \\ 0 & & \ddots & 1 \end{bmatrix}$$

(3) Q preserves angles and lengths

(a) $(Q\vec{v}) \cdot (Q\vec{w}) = \vec{v} \cdot \vec{w}$ ← angles

(b) $\|Q\vec{v}\| = \|\vec{v}\|$. ← lengths

<u>Proof</u> $(Q\vec{v}) \cdot (Q\vec{w}) = (Q\vec{v})^T Q\vec{w}$

$= \vec{v}^T (Q^T Q) \vec{w} = \vec{v}^T I \vec{w} = \vec{v}^T \vec{w}$

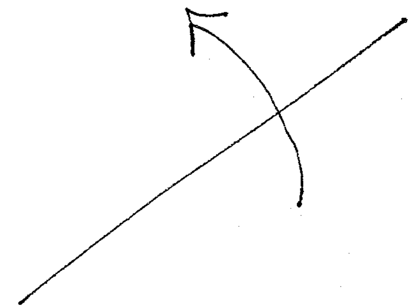
$= \vec{v} \cdot \vec{w}$

(b) follows from a since $\|Q\vec{v}\|^2$

$= Q\vec{v} \cdot Q\vec{v} = \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

Other examples later

- Reflections



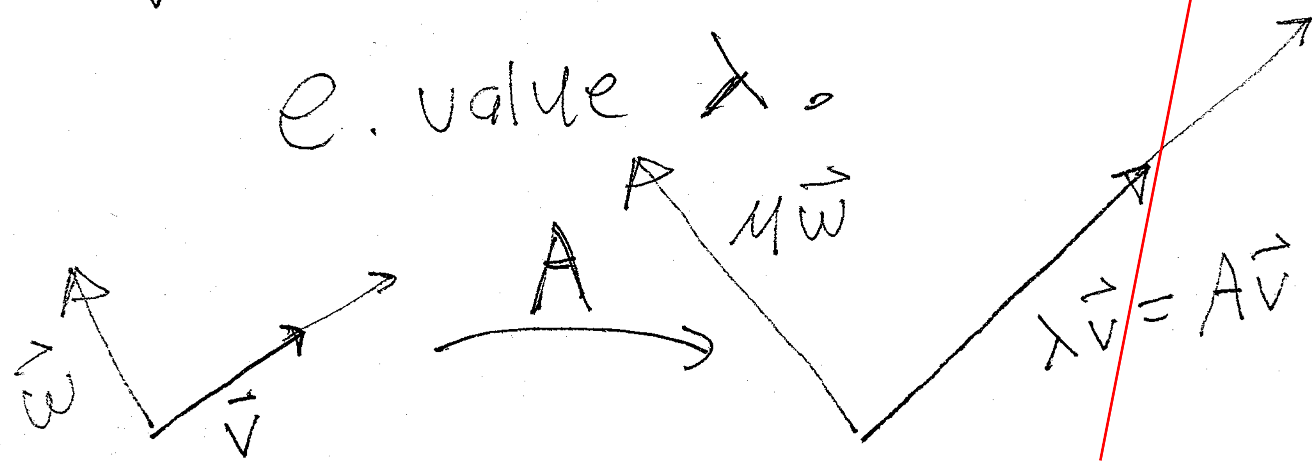
Eigenvector and Eigenvalues

Characteristic vectors
characteristic values.

$\boxed{\vec{v} \neq 0}$ $\quad A\vec{v} = \lambda \vec{v}$

$$\boxed{\begin{aligned} &A(\alpha\vec{v}) \\ &= \alpha A\vec{v} \\ &= \alpha\lambda\vec{v} \\ &= \lambda(\alpha\vec{v}) \end{aligned}}$$

$\vec{v}$ is e.vect with
e. value λ.



rescaling of e-vector
is e-vector

Where do they come from.

D.E.:

$$\frac{d\vec{x}}{dt} = A\vec{x}$$ ← Matrix ODE

Trial soln $\vec{x}(t) = e^{\lambda t}\vec{v}$

plugin

$$\frac{dx}{dt} = \lambda \cancel{e^{\lambda t}}\vec{v} = A\cancel{e^{\lambda t}}\vec{v}$$

$$\lambda\vec{v} = A\vec{v}$$ eigenvect eq.

eg11

$$\underbrace{\begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}_{\vec{v}_1} = \begin{bmatrix} 30 \\ 20 \end{bmatrix} = \underbrace{10}_{\lambda_1} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{v}_2} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \underbrace{5}_{\lambda_2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Computing by hand

$$A\vec{v} = \lambda \vec{v} \Rightarrow A\vec{v} - \lambda \vec{v} = 0$$

$$\Rightarrow (A\vec{v} - \lambda I \vec{v}) = 0$$

$$\Rightarrow (A - \lambda I)\vec{v} = 0. \quad \vec{v} \neq 0$$

$\Rightarrow$ $A - \lambda I$ is not invertible

$$\Rightarrow \det(A - \lambda I) = 0$$

degree n polynomial called the characteristic polynomial.

Recall $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

$$\det\left(\begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} 8-\lambda & 3 \\ 2 & 7-\lambda \end{pmatrix} = 0$$

evect next time

$$(8-\lambda)(7-\lambda) - 6 = 0$$

$$\lambda^2 - 15\lambda + 50 = 0$$

$$(\lambda-5)(\lambda-10) = 0$$

$$\boxed{\lambda = 5, 10}$$