# Eigen values and vectors

$$A\vec{v} = \lambda \vec{v} \qquad \vec{v} \neq \vec{0}$$

$A$ is $(n \times n)$ <u>square</u>

$$0 = p_A(\lambda) = \det(A - \lambda I) \quad \text{char. poly.}$$

it is a degree $n$ polynomial.

$$p_A(\lambda) = \lambda^n + \cdots$$

So $n$-roots, counting <u>multiplicities</u>

So $n$-eigenvalues, " "

**example**

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 = 0$$

$\lambda = 2$ with multiplicty 2.

"repeated root"

**<u>example from last time</u>**

$$A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$$

$$P_A(\lambda) = \det\begin{pmatrix} 8-\lambda & 3 \\ 2 & 7-\lambda \end{pmatrix} = \lambda^2 - 15\lambda + 50$$

$$= (\lambda-5)(\lambda-10) = 0$$

$\lambda = 5, 10$ <u>eigen values</u>

Next, eigen vectors.

$(A-\lambda I)\vec{v} = \vec{0}$ since $A\vec{v} = \lambda\vec{v} = \lambda I\vec{v}$

**Let $\lambda = 5$**, $\begin{bmatrix} 8-5 & 3 \\ 2 & 7-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

redundant

$$3v_1 + 3v_2 = 0$$
$$2v_1 + 2v_2 = 0$$

**Choice**

$v_1 = 1$
$v_2 = -1$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

some systems like MATLAB will return e-vectors of norm one, i.e. normalized

another choice $v_1 = \sqrt{2}$, $v_2 = -\sqrt{2}$

$\vec{v} = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \end{bmatrix}$

$\underline{\underline{\lambda = 10}}$

$$\begin{bmatrix} 8-10 & 3 \\ 2 & 7-10 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

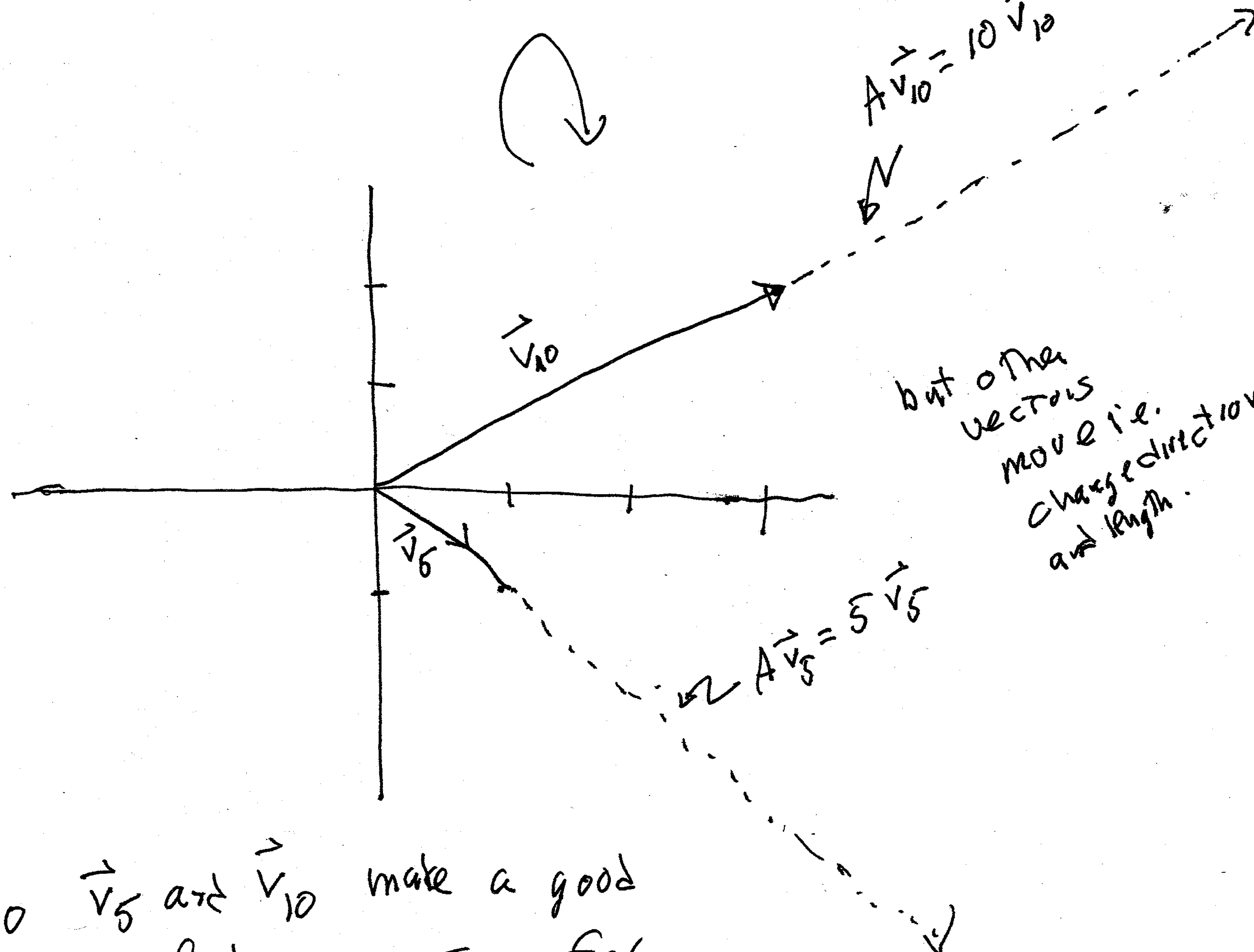
$-2\cancel{3}$

$$\left[\begin{aligned} -2v_1 + 3v_2 &= 0 \\ 2v_1 - 3v_2 &= 0 \end{aligned}\right.$$

<u>choice</u> $v_1 = 3 \quad v_2 = 2$

$$\boxed{\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

$A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$




but other vectors move i.e. change direction and length.

so $\vec{v}_5$ and $\vec{v}_{10}$ make a good choice of basis vectors for understanding how A acts.

Let's use $\vec{v}_5$ and $\vec{v}_{10}$ as a basis $\mathbb{R}^2$

$$\vec{w} = \alpha_1 \vec{v}_5 + \alpha_2 \vec{v}_{10}$$

$$\begin{aligned} A\vec{w} &= A(\alpha_1 \vec{v}_5 + \alpha_2 \vec{v}_{10}) \\ &= \alpha_1 A \vec{v}_5 + \alpha_2 A \vec{v}_{10} \\ &= (\alpha_1 5)\vec{v}_5 + (\alpha_2 10)\vec{v}_{10} \end{aligned}$$

So if coordinates of $\vec{w}$ are $\alpha_1, \alpha_2$

" " $A\vec{w}$ are $5\alpha_1, 10\alpha_2$.

A acts like $\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$

## General Situation

Assume $A\vec{v}_i = \lambda \vec{v}_i$ are the eigen-vectors with $\{\vec{v}_1, \ldots, \vec{v}_n\}$ a basis for $\mathbb{R}^n$

note: the collection of e-vectors don't always form a basis

if $\vec{w} = \sum \alpha_i \vec{v}_i$

$$A\vec{w} = \sum \alpha_i \lambda_i \vec{v}_i$$

So in other words, if $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$

and $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ when $\vec{w} = \sum \alpha_i \vec{v}_i$

coord of $\vec{w}$ is the basis $\mathcal{B}$

Then $\left[A\vec{w}\right]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \alpha_1 \\ \lambda_2 \alpha_2 \\ \vdots \\ \lambda_n \alpha_n \end{bmatrix}$

<u>So</u> $\left[A\vec{w}\right]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} \left[\vec{w}\right]_{\mathcal{B}}$

$\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Another version

Assume $A$ has eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ which lin. ind. write

$$\mathbb{X} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \quad \boxed{\text{col vect}}$$

$$\Rightarrow \underbrace{\mathbb{X}^{-1} A \mathbb{X}}_{} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

conjugation or similarity.

which diagonalizes the matrix

it represents a change of coordinates into the eigen basis

Proof

$$
\begin{aligned}
A\underline{X} &= A\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \\
&= \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} \\
&= \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{bmatrix} = \underline{X}\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)
\end{aligned}
$$

so $A\mathbb{X} = \mathbb{X}\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)$

Col. of $\mathbb{X}$ are lin ind., so $\mathbb{X}$ is invertible.

so $\mathbb{X}^{-1} A \mathbb{X} = \mathbb{X}^{-1}\mathbb{X}\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)$

$= \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

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example $A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$ $\quad \mathbb{X} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

$\mathrm{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 5 & 0 \\ 0 & \boxed{10} \end{bmatrix}$

make sure columns correspond to position of eigenvalue

Then $\mathbb{X}^{-1} A \mathbb{X} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$

# Complex e.values and e.vectors also occur

$$A = \begin{bmatrix} -1 & -2 \\ 5 & 5 \end{bmatrix}$$

$$P_A(\lambda) = \det \begin{bmatrix} -1-\lambda & -2 \\ 5 & 5-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 5$$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm 2i}{2} = \boxed{2 \pm i}$$

complex conjugate pair when matrix is real

$$\underline{\underline{\lambda = 2+i}} \quad \begin{bmatrix} -1-(2+i) & -2 \\ 5 & 5-(2+i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} (-3-i)v_1 - 2v_2 &= 0 \\ 5v_1 + (3-i)v_2 &= 0 \end{aligned}$$

redundant once is complex multiple of the other

$$(3+i)v_1 + 2v_2 = 0$$

$$\boxed{\begin{aligned} v_2 &= 3+i \\ v_1 &= -2 \end{aligned}} \qquad \vec{v} = \begin{bmatrix} -2 \\ 3+i \end{bmatrix}$$

$\boxed{\text{acts on } \mathbb{C}^2}$

$\lambda = 2 - i$

$-1-(2-i)v_1 + -2v_2 = 0$

$(-3+i)v_1 - 2v_2 = 0$

$v_1 = -2$

$v_2 = 3 - i$

$\lambda = 2 - i \quad \vec{V} = \begin{bmatrix} -2 \\ 3-i \end{bmatrix}$

$\lambda = 2 + i \quad \vec{V_1} = \begin{bmatrix} -2 \\ 3+i \end{bmatrix}$

$\alpha + i\beta$

$\alpha - i\beta$

are complex conjugates