Eigenvalues + vectors

$X = [\vec{v}_1 \dots \vec{v}_n]$

Diagonalization Theorem

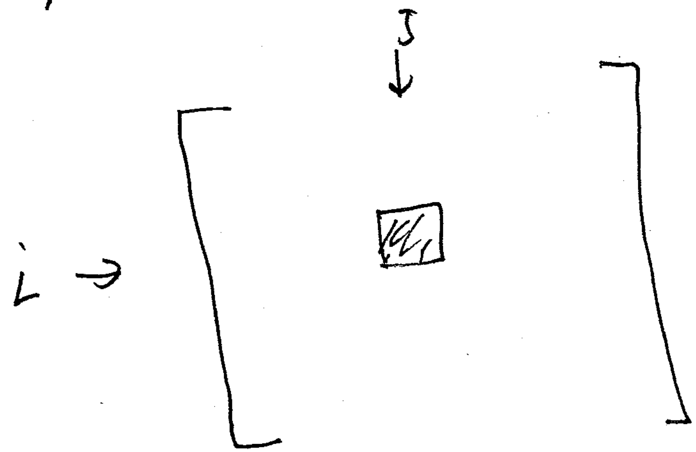
Assume A is $(n \times n)$ and has eigenvector, eigenval pairs $(\lambda_i, \vec{v}_i)$ so that $\{\vec{v}_1, \dots, \vec{v}_n\}$ are $\boxed{\text{linearly ind.}}$ (so they are a basis of $\mathbb{R}^n$)

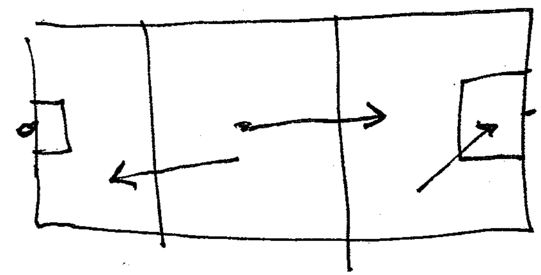
then $X^{-1} A X = \text{diag}(\lambda_1, \dots, \lambda_n) = \Lambda$

<u>example</u> $A = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$ $X = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$

$$\Lambda = X^{-1} A X$$

SAY A is a MARKOV Transition matrix





A_{ij} = probability of a transition from state i to state j

$AAA \cdots A = A^n$ is the process after n steps.

3

$$\boxed{A = X \Lambda X^{-1}}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

$$A^2 = X \Lambda X^{-1} X \Lambda X^{-1}$$

$$= X \Lambda^2 X^{-1}$$

$$\vdots$$

$$A^k = X \Lambda^k X^{-1} = X \operatorname{diag}(\lambda_1^k \ldots \lambda_n^k) X^{-1}$$

In general two matrices A and B are similar if $B = C^{-1} A C$ for an <u>invertible</u> C.

<u>Th'm</u> $\Rightarrow$ (1) A and B have the same eigenvalues (spectrum = set of eigen values).

(2) If $(\lambda_i, \vec{v}_i)$ is an eigenval, vect pair for B

$\Rightarrow$ $(\lambda_i, C\vec{v}_i)$ is an e.val, vect pair for A.

(6)

~~oof~~ $P_B(\lambda) = |B - \lambda I| \leftarrow \det$

$= |C^{-1}AC - \lambda I|$

$= |C^{-1}AC - \lambda \overset{\text{trick!}}{C^{-1} I C}|$

$= |C^{-1}(A - \lambda I)C|$

$= |C^{-1}|\,|A - \lambda I|\,|C|$

$= |A - \lambda I|$

$= p_A(\lambda) \Rightarrow A$ and B have The same charpoly so same spectrum.

$$\boxed{\begin{aligned} C^{-1}C &= I \\ |C^{-1}C| &= |I| = 1 \\ |C^{-1}||C| &= 1 \end{aligned}}$$

For (b) $\quad B\vec{v}_L = \lambda_L \vec{v}_L$

$$\lambda_L \vec{v}_L = B\vec{v}_L = C^{-1}AC\vec{v}_L$$

$$C(\lambda_L \vec{v}_L) = AC\vec{v}_L$$

$$\| $$

$$\lambda_L (C\vec{v}_L) = A(C\vec{v}_L)$$

so if $\vec{w}_L = C\vec{v}_L \Rightarrow \lambda_L \vec{w}_L = A\vec{w}_L$

Solving Linear Matrix DE via Diagonalization.

Liner Ma trix DE $\Longleftrightarrow$ $n \times n$ coupled system of Linear DE.

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{dx_1}{dt} = 8\underline{\underline{x_1}} + 3\underline{\underline{x_2}}$$

$$\frac{dx_2}{dt} = 2\underline{\underline{x_1}} + 7\underline{\underline{x_2}}$$

General set up: $\vec{x} \in \mathbb{R}^n$, A is $(n \times n)$

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

<u>Assume</u> A is diagonizable

$$\boxed{\begin{aligned} X^{-1} A X &= \Lambda \\ A &= X \Lambda X^{-1} \end{aligned}}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} = X \Lambda X^{-1} \vec{x}$$

$$\frac{d(X^{-1}\vec{x})}{dt} = X^{-1}\frac{d\vec{x}}{dt} = \Lambda (X^{-1}\vec{x})$$

Let $\underbrace{\vec{y} = X^{-1}\vec{x}}_{\text{new coordinates}}$

so $$\frac{d\vec{y}}{dt} = \Lambda \vec{y} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)\vec{y}$$

$$\left.\begin{aligned} \frac{dy_1}{dt} &= \lambda_1 y_1 \\ \frac{dy_2}{dt} &= \lambda_2 y_2 \\ &\vdots \\ \frac{dy_n}{dt} &= \lambda_n y_n \end{aligned}\right\} \text{ decoupled}$$

each equation

$$\boxed{\begin{aligned} \frac{dy}{dt} &= \lambda y \\ y(t) &= y(0)e^{\lambda t} \end{aligned}}$$

SOLN: $y_j(t) = y_j(0)e^{\lambda_j t}$

for $j = 1, \ldots, n$.

In matrix form.

$$\vec{y}(t) = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \\ \vdots \\ y_n(0) \end{bmatrix}$$

$$\vec{y}(t) = \mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})\, \vec{y}(0)$$

Soln in $\vec{y}(t)$. We need soln. for $\underline{\underline{\vec{x}(t)}}$.

$$\vec{y} = X^{-1}\vec{x} \rightarrow \vec{x} = X\vec{y}$$

$$X\vec{y}(t) = X\,\mathrm{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})\,\vec{y}(0)$$

$$\boxed{x(t) = X\,\mathrm{diag}(e^{\lambda_1 t} \ldots e^{\lambda_n t})\left(X^{-1}\vec{x}(0)\right)}$$

Only depends on spectrum

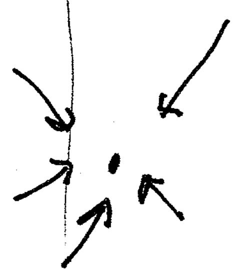
Stability – what happens to $x(t)$ as $t \to \infty$.

If all $\lambda_i < 0 \Rightarrow x(t) \to 0$	some $\lambda_i > 0$ $x(t) \to \infty$

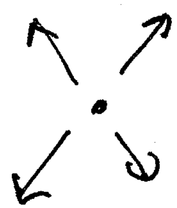
$$\boxed{e^{\lambda t} \begin{array}{ll} \to 0 & \lambda < 0 \\ \to \infty & \lambda > 0 \\ = 1 & \lambda = 0 \end{array}}$$

Assuming eigenvalues are real

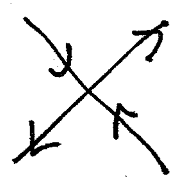
In $\mathbb{R}^2$ assuming eigenvalues are real



stable $\lambda_1, \lambda_2 < 0$ (sink, attractor)



unstable $\lambda_1, \lambda_2 > 0$ (source, repeller)



unstable $\lambda_1 < 0$, $\lambda_2 > 0$ (saddle)