A big idea is Data Science, DSP, Image processing, Scientific computation...

\[ \hat{f} \]

is a data vector, signal, image, function

\[ \hat{f} \] is in \( V \), a vector space.

- Find an orthonormal basis \( \hat{z}_1, \hat{z}_2, \ldots \) for \( V \) that encodes some properties of interest.

- Expand \( \hat{f} \) in terms of the basis

\[ \hat{f} = d_1 \hat{z}_1 + d_2 \hat{z}_2 + \ldots \]

\[ \text{New} \]

1. \( d_j \) is the amount of \( \hat{f} \) in \( \hat{z}_j \)

2. Truncating the expansion

\[ \hat{f}(x) = d_1 \hat{z}_1 + \ldots + d_k \hat{z}_k \]

stores an efficient, lower dimension version of \( \hat{f} \) which still encodes essential information.
Where do we get the orthonormal basis

- Eigenvectors of Hermitian matrix (operator)
- Gram-Schmidt process on another basis
- Science, ...

Example 1: Let \( V \) be all polynomials with real coefficients defined on \([-1, 1]\). Then

\[ \exists \{1, t, t^2, \ldots\} \text{ is a basis} \]

Since any polynomial can be written

\[ p(t) = q_0 \cdot 1 + q_1 \cdot t + \ldots + q_n \cdot t^n \]

(That is the definition of a polynomial.)

Now put the inner product on \( V \)

\[ \langle p, z \rangle = \sum_{-1}^{1} p(t)z(t)\, dt \]
Then using Gram-Schmidt on the given basis \( \frac{1}{\sqrt{2}} \), \( \frac{1}{\sqrt{2}} + i \), \( \frac{1}{\sqrt{2}} - i \), yields an orthonormal basis for \( V \):

\[
\varphi_0(t) = \frac{1}{\sqrt{2}}, \quad \varphi_1(t) = \frac{\sqrt{3}}{\sqrt{2}} t, \quad \varphi_2(t) = \frac{1}{2\sqrt{2}} (3t^2 - 1), \ldots
\]

These are called the Legendre polynomials.

**Example 2:** Let \( V = L^2([-\pi, \pi]) \) with the Hermitian inner product \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} \, dt \).

The Fourier basis is orthonormal:

\[
\frac{e^{-inx}}{\sqrt{2\pi}}, \frac{e^{-2ix}}{\sqrt{2\pi}}, \frac{e^{-ix}}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{e^{ix}}{\sqrt{2\pi}}, \frac{e^{2ix}}{\sqrt{2\pi}}, \ldots
\]

Note that these are indexed by all integers, not just positive ones and recall

\[
e^{inx} = \cos nx + i \sin nx
\]
Let's check they are orthonormal.

If \( m \neq n \)

\[
\left< \frac{e^{imx}}{\sqrt{2\pi}}, \frac{e^{inx}}{\sqrt{2\pi}} \right> = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx} \, dx
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx = \frac{1}{2\pi} \left. \frac{e^{i(n-m)x}}{i(n-m)} \right|_{-\pi}^{\pi}
\]

\[
= \frac{1}{2\pi i(n-m)} \left[ e^{i(n-m)\pi} - e^{-i(n-m)\pi} \right] = 0
\]

\[
\text{when } (n-m) \text{ is odd}
\]

\[
= -1 - 1\\ 1 - 1
\]

\[
\text{when } (n-m) \text{ is even}
\]

and \( \left< \frac{e^{imx}}{\sqrt{2\pi}}, \frac{e^{imx}}{\sqrt{2\pi}} \right> = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{imx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \, dx = \frac{2\pi}{2\pi} = 1 \)

This shows o.n., showing it is a basis is harder.
Example 3: The discrete Fourier basis \( \mathcal{F} \) is an orthonormal basis for \( \mathbb{C}^n \) obtained by discretizing the usual Fourier basis. It is the subject of the next few lectures.

Recall that we want to use the orthonormal bases to give coordinates to vectors. The next theorem says how to do this.

Theorem: If \( \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n \) is an orthonormal basis w.r.t. a hermitian inner product \( \langle \cdot, \cdot \rangle \), then

\[
\vec{v} = \sum \alpha_i \vec{e}_i \quad \text{with each } \alpha_i = \langle \vec{v}, \vec{e}_i \rangle
\]
Proof: We have since \( \{ \vec{z}_1, \ldots, \vec{z} \} \) is a basis that for some \( d_1 \):

\[
\vec{v} = d_1 \vec{z}_1 + d_2 \vec{z}_2 + \ldots
\]

then using the linearity of the Hermitian inner product:

\[
\langle \vec{z}_k, \vec{v} \rangle = \langle \vec{z}_k, d_1 \vec{z}_1 \rangle + \langle \vec{z}_k, d_2 \vec{z}_2 \rangle + \ldots
\]

\[
+ \langle \vec{z}_k, d_k \vec{z}_k \rangle + \ldots
\]

\[
= d_1 \langle \vec{z}_k, \vec{z}_1 \rangle + d_2 \langle \vec{z}_k, \vec{z}_2 \rangle + \ldots
\]

\[
+ d_k \langle \vec{z}_k, \vec{z}_k \rangle
\]

\[
= 0 + 0 + \ldots + d_k
\]

Since \( \{ \vec{z}_1, \vec{z}_2, \ldots, \vec{z} \} \) is an o.n. basis.
Example: For the Fourier basis for \( L^2[-\pi, \pi] \)

If \( f(\theta) = \sum \alpha_n e^{int} \)

\[ \alpha_n = \langle e^{int}, f(\theta) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{int} f(\theta) \, d\theta \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(\theta) e^{-int} \, d\theta \] (usual to write it in this order).

Now let \( f(\theta) = \begin{cases} 1 & \text{when } 1+1 \leq |\theta| < 1+1/2 \\ 0 & \text{when } \frac{3\pi}{2} < |\theta| \leq \pi \end{cases} \)

We want to express \( f \) is the Fourier basis, i.e. find its Fourier expansion.
\[ \alpha_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} f(t) e^{-i\pi t^2} dt = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} e^{-\pi t^2} dt \]

\[ = -\frac{1}{\sqrt{2\pi}} \sin \left( \frac{n\pi}{2} \right) \]

\[ = \frac{2\sin \left( \frac{n\pi}{2} \right)}{\sqrt{2\pi}} \text{ using Euler's formula.} \]

Notice this n, so \( n \neq 0 \), do \( n = 0 \) separately.

\[ \alpha_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} e^{0it} dt = \frac{\pi}{\sqrt{2\pi}} \]
So \( f(\frac{1}{\pi}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin(n\pi x)}{T} \cdot e^{-\frac{x^2}{2\sigma^2}} dx \).  

Using the fact that the non-zero terms come in pairs \( n, -n \), so adding by pairs gives \( \sin(n\pi x) = \frac{1}{2i} \left( e^{in\pi x} - e^{-in\pi x} \right) \).  

Thus we have the expansion \( e^{-\frac{x^2}{2\sigma^2}} \cos nt = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2 \sin(n\pi x)}{n\pi} \cos nt \).  

Therefore, \( \sum_{n=1}^{\infty} \frac{2 \sin(n\pi x)}{n\pi} \cos nt = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2 \sin(n\pi x)}{n\pi} \cos nt \).
There are two more fundamental facts about a unitary basis.

The Pythagorean Theorem: If \( z, z', \ldots, z_n \) is an orthonormal set, then for any \( \sum_{i=1}^{n} |z_i|^2 = \sum_{i=1}^{n} |z_i|^2 \).

Proof: \[ \langle z_i, z_i \rangle = \langle z_i, z_i \rangle + \langle z_i, z_i \rangle + \cdots + \langle z_i, z_i \rangle = \sum_{i=1}^{n} |z_i|^2 \]
In signal processing and compression, you often want a representation of the signal that only uses the lower, more significant harmonics or frequencies. What should this be?

The next theorem says that the least squares best fit comes from truncating the D.N. expansion.

So, \( f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{int} \)

The best order \( N \) fitting trig polynomial is:

\[
\sum_{n=-N}^{N} \alpha_n \frac{e^{int}}{n2\pi i}
\]
Theorem: Say $\exists q_1, \ldots, q_3$ is an orthonormal set

and

and let

with equality only when $V = \sum_{i=1}^{N} q_i z_i$

Thus

Thus
Proof we compute both sides using the Pythagorean Theorem 

\[
\| \vec{v} - \vec{w} \|^2 = \sum_{i=1}^{N} a_i \vec{z}_i - \sum_{i=1}^{N} \alpha_i \vec{z}_i \| \| \sum_{i=1}^{N} \alpha_i \vec{z}_i \|^2 = \sum_{i=1}^{M} \alpha_i \vec{z}_i \|^2 \]

\[
\| \vec{v} - \vec{w} \|^2 = \sum_{i=1}^{N} a_i \vec{z}_i - \sum_{i=1}^{N} \beta_i \vec{z}_i \| \| \sum_{i=1}^{N} \beta_i \vec{z}_i \|^2 = \sum_{i=1}^{M} \beta_i \vec{z}_i \|^2 \]

\[
\| \vec{v} - \vec{w} \|^2 = \sum_{i=1}^{N} a_i - \sum_{i=1}^{N} \beta_i \| \| \sum_{i=1}^{N} \beta_i \|^2 = \sum_{i=1}^{M} \beta_i \vec{z}_i \|^2 \]

\[
\sum_{i=1}^{N} a_i - \sum_{i=1}^{N} \beta_i \| \| \sum_{i=1}^{N} \beta_i \|^2 = \sum_{i=1}^{M} \beta_i \vec{z}_i \|^2 \]

So \[
\| \vec{v} - \vec{w} \|^2 - \| \vec{v} - \vec{w} \|^2 = \sum_{i=1}^{M} \beta_i \vec{z}_i \|^2 \geq 0
\]

and is equal to zero only when \( \alpha_i = \beta_i \) for \( i = 1, \ldots, N \)

or when \( \vec{w} = \vec{v}/\mu \).