The Combinatorics of Rational Parking Functions and the Shuffle Conjectures

Let $\mathcal{D}_{m,n}$ be the set of the set of paths consisting of unit north and east steps that start at $(0,0)$ and end at (m,n) which stay on or above the line $y = \frac{n}{m}x$. We let $\mathcal{P}_{m.n}$ denote the set of rational parking functions which consists of a path $D \in \mathcal{D}_{m,n}$ where we label the north steps of D with the numbers $1, 2, \ldots, n$ such that the labels increase, reading from bottom to top, in each column. There is now a rich theory of the combinatorics of rational parking functions and their connections with representation theory and algebraic geometry. For example, the Shuffle Conjecture of Haglund, Haiman, Loehr, Remmel, and Ulyanov gave the quasisymmetric function expansion of the Frobenius image of the ring of Diagonal Harmonics in terms of certain statistics on parking functions in $P_{n,n}$. This conjecture was later refined by Haglund, Morse, and Zabrocki which they called the Compositional Shuffle Conjecture. The Compositional Shuffle Conjecture was proved by Carlsson and Melit in 2015. When n and m are co-prime Gorski and Negut used the work on the Elliptic Hall Algebra of Shiffmann-Vasserto and the work on Springer Fibers of Hikita to formulate an extension of the Shuffle Conjecture to rational parking functions which is now called the Rational Shuffle Conjecture. Later Bergeron, Garsia, Leven, and Xin extended the Compositional Shuffle Conjecture to the setting of rational parking functions. The Rational Shuffle conjecture was recently proved by Melit. In a different direction, Armstrong, Loehr, Warrington, Rhoades, Williams and other have studied an action of the symmetric group on rational parking functions when m and n are relatively prime and developed a rich theory which is now called Coxeter-Catalan Combinatorics. In this talk, I will survey several of these topics.

Jeff Remmel is a Distinguished Professor of Mathematics at the University of California, San Deigo. His research interests include Combinatorics, Logic, Theoretical Computer Science, and Hybrid Control. He has published over 300 papers and graduated over 30 Ph.D. students.