Eg// last time

$$\iota_{S^n}: S^n \hookrightarrow \mathbb{R}^{n+1} - \{0\}$$

inclusion

induces an isomorphism on the fund group.

<u>DEF</u>: $r: X \to A$ is a retraction if

$r|_A = id_A$ OR

$r \circ \iota_A = id_A$

__DEF:__ $A \subseteq X$ is a __deformation retract__ if id_X is homotopic to a map that collapse X into A and each pt in A is fixed during the homotopy:

More precisely, $\exists$ cont $H: X \times I \to X$ such that

(a) $H(x,0) = x$

(b) $H(x,1) \in A$

(c) $H(a,t) = a \quad \forall a \in A$

- H is called a deformation retraction

~~$H_0 \simeq id_X$~~

- Let $r(x) = H(x,1)$ then
  $r : X \to A$ by (b)
  and $r(a) = a \quad \forall a \in A$ by (c)
  So r is a retract $X \to A$
- $H : id_X \simeq i_A \circ r$

NOTE! Some books call this a strong deformation retract

Example S^n and $\mathbb{R}^{n+1} - \{0\}$

$S^n \subseteq \mathbb{R}^{n+1} - \{0\}$

__Thm__ If A is a def retract ~~of~~ of X pick $a_0 \in A \Rightarrow i_A : A \hookrightarrow X$

$$(i_A)_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$$

is an isomorphism.

__Proof__: $i_A \circ r \simeq id_X$ by ~~def~~ deformation retract

$r \circ i_A = id_A$ by retract

where r is defined as above.

These homotopies fix $a_0 \in A$ which

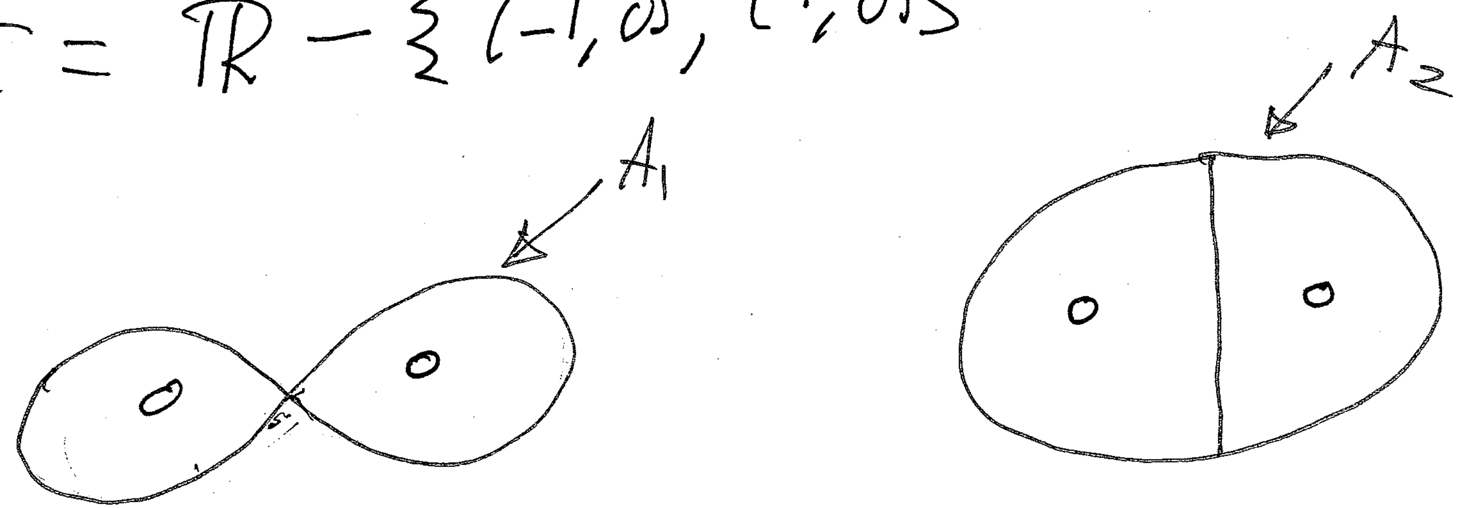
$\Rightarrow$ hit with "$*$" on π_1.

- $X = \mathbb{R}^n \quad A = \{\vec{0}\}$

$\Rightarrow A$ is a def retract of X 

So $\pi_1(X, \vec{0}) = \{e\}$.

- $X = \mathbb{R}^2 - \{(-1,0), (1,0)\}$



Deformation retract isn't an equiv. relation. $\pi_1(X) =$ free group on 2 symbols.

DEF: $f: X \to Y \quad g: Y \to X$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

continuous. are called a homotopy equivalence if $g \circ f \simeq id_X \quad f \circ g \simeq id_Y$.

In this case, X and Y are said to be homotopy equivalent or of the same homotopy type OR homotopic.

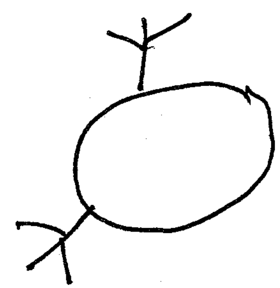
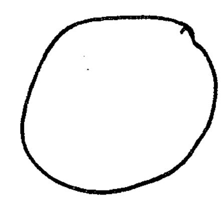
RK: Homotopy equiv is an equiv relation on topological spaces.

eg: (1) $A \subseteq X$ def retract $\Rightarrow A \simeq X$.

(2) $\{0\} \subseteq \mathbb{R}^n$ is a def retract

so $\{0\} \simeq \mathbb{R}^n \ \forall n \Rightarrow$

$\mathbb{R}^n \simeq \mathbb{R}^m$.

(3)  $\simeq$ 

Thm $X \simeq Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$

Converse false

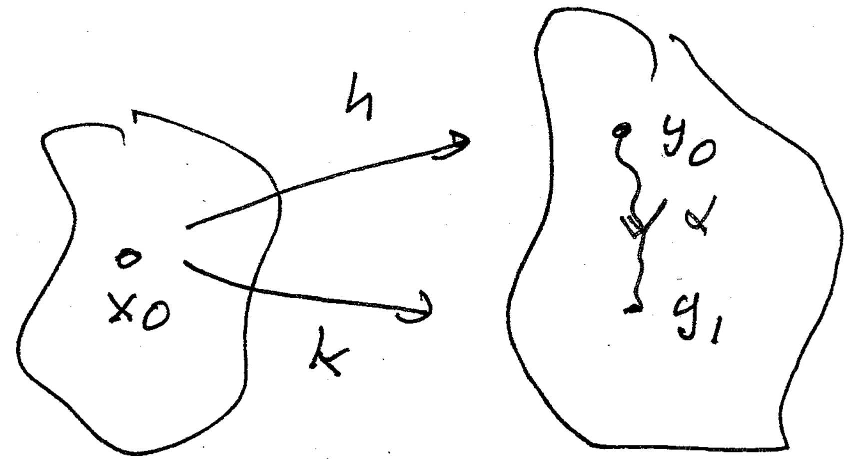
$X = S^2 \qquad Y = \mathbb{R}$

$\pi_1(X) \cong \{e\} \cong \pi_1(\mathbb{R})$

but $S^2 \not\cong \mathbb{R}$.

__Lemma:__ $h, k : X \to Y$, continuous

$h(x_0) = y_0$
$k(x_0) = y_1$



$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\ & \pi_1(Y, y_1) \end{tikzcd}$$

where $H : \underset{h \simeq k}{\cancel{\ }}$ and $\alpha(t) = H(x_0, t)$. $\blacksquare$

Proof: If f is a loop in X based at x_0 we need

$$k_*[f] = \hat{\alpha} \circ h_*[f]$$

or

$$[k \circ f] = [\alpha^{-1}] * [h \circ f] * [\alpha]$$

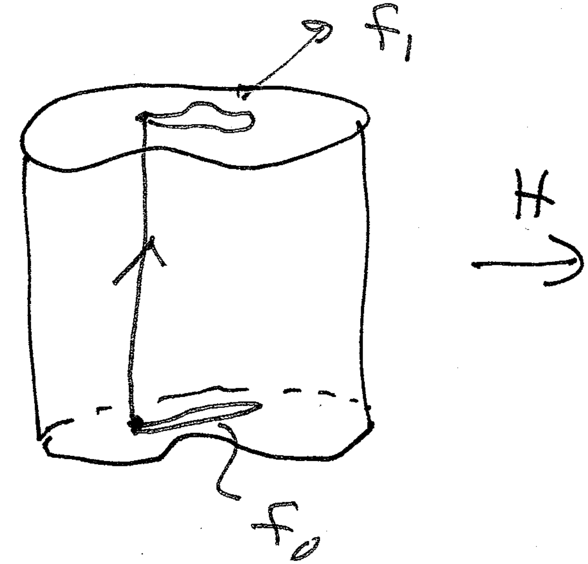
or

$$[\alpha] * [k \circ f] = [h \circ f] * [\alpha]$$

which we prove.

Define two loops in $X \times I$ as

$$f_0(x) = (f(x), 0)$$
$$f_1(x) = (f(x), 1)$$
$$c(t) = (x_0, t)$$

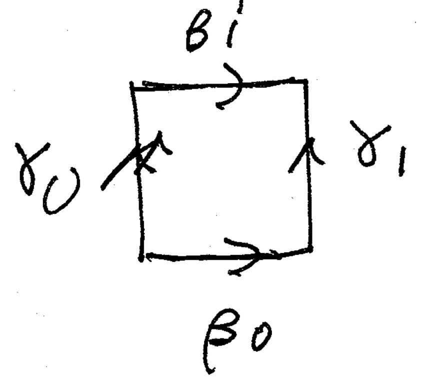


Then $H \circ f_0 = h \circ f$
$H \circ f_1 = k \circ f$
$H \circ c = \alpha$.

Also define $F : I \times I \to X \times I$

by $F(\cancel{x}, t) = (f(x), t)$

and paths in $I \times I$ as

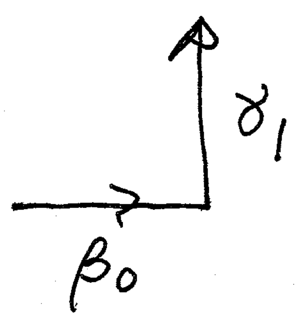


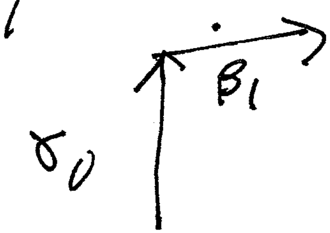
$\beta_0(s) = (s, 0)$ $\quad \beta_1(s) = (s, 1)$

$\gamma_0(t) = (0, t)$ $\quad \gamma_1(t) = (1, t)$

$F \circ \beta_0 = f_0$ $\qquad F \circ \gamma_0 = F \circ \gamma_1 = c$

$F \circ \beta_1 = f_1$





Since $I \times I$ is convex

$$\beta_0 * \gamma_1 \simeq_p \gamma_0 * \beta_1 \text{ in } I \times I$$

by a homotopy we call G.

Then $F \circ G : f_0 * c \simeq_p c \circ f_1$

So $H \circ F \circ G$ is a path homotopy, between

$$H(f_0 * c) = (H \circ f_0) * (H \circ c)$$
$$= (h \circ f) * \alpha.$$

$$H(c \circ f_1) = (H \circ c) * (H \circ f_1)$$
$$= \alpha * (k \circ f)$$

__COR__ $h, k : X \to Y$ cont, homotopic then

$\Rightarrow$ h_* is injective, surjective, triv

$\Leftrightarrow$ k_* has the same property.

__COR__ $h : X \to Y$ cont, null homotopic

$\Rightarrow$ h_* is the triv homomorphism.

__Thm__ $f : X \to Y$ is a homotopy equivalence

$f(x_0) = y_0 \Rightarrow f_* : \pi_1(X, x_0) \to$

$\pi_1(Y, y_0)$ is an isomorphism.