Every last time

\[ \Sigma^n : S^n \to \mathbb{R}^{n+1} - S^n \]

inclusion

Induces an isomorphism on the fundamental group.

**DEF:** \( r \) : \( X \to A \) is a retraction if

\[ r | A = \text{id}_A \quad \text{or} \quad r | \partial A = \text{id}_A \]

\[ r \circ i_A = \text{id}_A \]
DEF: \( A \subseteq X \) is a deformation retract if \( \text{id}_X \) is homotopic to a map \( \varphi \) that collapses \( X \) into \( A \) and each \( p \) in \( A \) is fixed during homotopy.

More precisely, \( \exists \) \( \text{cont} \ H : X \times I \rightarrow X \) such that

1. \( H(x,0) = x \)
2. \( H(x,1) \in A \)
3. \( H(p,t) = p \) \( \forall p \in A \)

\( H \) is called a deformation retraction.
Let \( r(x) = H(x,t) \) then
\[
\begin{align*}
\text{r: } & X \to A \text{ by (b)} \\
& \forall x \in X \text{ by (c)} \\
& a \cdot r(a) = a \quad \forall a \in A
\end{align*}
\]
So \( r \) is a retract \( X \to A \).

Hence \( \text{id}_X = 1_A \circ r \).

**Note:** Some books call this a 
strong deformation retract

**Example:** \( S^n \) and \( \mathbb{R}^{n+1} \setminus \{0\} \)
\[
S^n \subset \mathbb{R}^{n+1} \setminus \{0\}
\]
Theorem: If $A$ is a def retract of $X$, pick $a_0 \in A \Rightarrow L_A: A \rightarrow X$

$(L_A)_*: \pi_A(A, a_0) \rightarrow \pi_1(X, a_0)$

is an isomorphism.

Proof: $L_A \circ r = \text{id}_A$ by retract

$r \circ L_A = \text{id}_A$ by retract

where $r$ is defined as above.

These homotopies fix $a_0 \in A$ which

$\Rightarrow$ hit with "x" on $\pi_1(X, a_0)$.
\[ X = \mathbb{R}^n \quad A = \mathbb{D}^3 \]

\[ \Rightarrow A \text{ is a def retract of } X \]

So \( \pi_1 (X, \overline{0}) = \mathbb{Z} \mathbb{Z} \)

\[ X = \mathbb{R}^2 - \mathbb{Z} \{(1,0), (-1,0)\} \]

Deformation retract isn't an equivalence relation. \( \pi_1 (X) = \text{free group on 2 symbols} \).
DEF: A homotopy is an equivalence or homotopy type if $\pi_1 = \pi_0 = \emptyset$. In this case, $X$ and $Y$ are said to be homotopic.

R.K.: Homotopy is an equivalence relation on topological spaces.
\[ \text{If } x = 9 \Rightarrow \text{ Then } \exists (x) \leq 11, (9) \]

\[ (3) \]

\[ \text{If } n \in \mathbb{R} \quad \text{Then} \quad \exists \forall \phi \in \mathbb{R} \quad 0 \leq 3 \phi = \text{?} \]

\[ \text{If } 303 = \text{?} \quad \text{Is a test vector} \]

\[ \text{If } n \in \mathbb{R} \quad \text{Then} \quad \exists \forall \text{ vector } \]

\[ \text{If } A \in \text{ vector} \Rightarrow A = \]
Converse False

\[ x = S^2 \quad y = R \]

\[ \Pi_1 (x) \equiv \sum_{e \in \text{even}} \frac{\Pi_1 (\text{1R})}{2} \]

but \( S^2 \neq \text{IR} \).
Lemma: \( h, k : X \to Y \), continuous

\[
h(x_0) = y_0 \\
k(x_0) = y_1
\]

\[
P_1(X, x_0) \xrightarrow{h \ast k} P_1(y, y_0) \downarrow \chi
\]

\[
\chi
\]

\[
\tau
\]

\[
P_1(y, y_1) \xrightarrow{k \ast k}
\]

Where \( H : \mathbb{R}^2 \to \mathbb{R}^2 \) and \( d(x, y) = \|x - y\| \).
Proof! If \( f \) is a loop in \( X \) based at \( x_0 \) we need

\[ k \cdot [f] = \hat{x} \circ h_k \cdot [f] \]

or

\[ [k \circ f] = \sum_{\alpha} [\hat{x}] \cdot [h_\circ f] \cdot [\alpha] \]

or

\[ \sum_{\alpha} [\alpha] \cdot [k \circ f] = \sum_{\alpha} [h_\circ f] \cdot [\alpha] \]

which we prove.
Define two loops in $\mathbb{S}^1 \times T$ as:

$C(f)$

$F(f) = \{ (f(x), y) \}$

For $f_0, f_1 \in \pi_1$, we have

$H_0 f_0 = \text{hof}_f f_0$  $\Rightarrow$  $f_0 \circ x = x$

Pen
Also define \( F : I \times I \to X \times I \)

by \( F(x, t) = (f(x),\pm t) \)

and paths in \( I \times I \) as

\[
\begin{align*}
\beta_0(s) &= (s,0) & \beta_1(1s) &= (s,1) \\
\delta_0(\pm) &= (0,\pm) & \delta_1(\pm) &= (1,\pm)
\end{align*}
\]

\[
\begin{align*}
F \circ \beta_0 &= \gamma_0 \\
F \circ \delta_0 &= F_0 \gamma_1 = C \\
F \circ \beta_1 &= \gamma_1
\end{align*}
\]
Since I × I is convex

\[ \beta_0 \star \beta, \quad \frac{\cdot}{p} \beta_0 \star \beta, \quad \text{in} \quad I \times I, \]

by a homotopy we call \[ \beta G. \]

Then, \( F \circ G : f_0 \times c \cong c \circ f_1 \)

So \( H \circ F \circ G \) is a path homotopy between

\[ H (f_0 \times c) = (H \circ f_0) \star (H \circ c) \]

\[ = (h \circ f) \star \alpha. \]

\[ H (c \circ f_1) = (H \circ c) \star (H \circ f_1) \]

\[ = \alpha \star \beta (k \circ f). \]
Cor: \ h, k : X \to Y \text{ cont, homotopic then }\n
\implies h_k \text{ is injective, surjective, trivial }\n\implies K_k \text{ has the same property.}\n
Cor: \ h' : X \to Y \text{ cont, null homotopic }\n\implies h'_k \text{ is the trivial homomorphism.}\n
\exists m \quad \text{f : X \to Y is a homotopy equivalence}\n
\Rightarrow f(x_0) = y_0 \implies f_k^* : \pi_1(X, x_0) \to \pi_1(Y, y_0) \text{ is an isomorphism.}