Theorem: $f:(X, x_0) \to (Y, y_0)$ is homotopy equivalence $\Rightarrow$ $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

Proof Let $g: Y \to X$ be the homotopy inverse. Consider

$$(X, x_0) \xrightarrow{f} (Y, y_0) \xrightarrow{g} (X, x_1) \xrightarrow{f} (Y, y_1)$$

induces

$$\pi_1(X, x_0) \xrightarrow{(f_{x_0})_*} \pi_1(Y, y_0) \xrightarrow{g_*} \pi_1(X, x_1) \xrightarrow{(f_{x_1})_*} \pi_1(Y, y_1).$$

$g \circ f : (X, x_0) \to (X, x_1)$ is homotopic to Id_X. By a previous lemma $\exists$ path α in X $x_0 \xrightarrow{\alpha} x_1$ with

$$(g \circ f_{x_0})_* = \hat{\alpha} \circ (\mathrm{Id}_X)_* = \hat{\alpha} \quad (1)$$

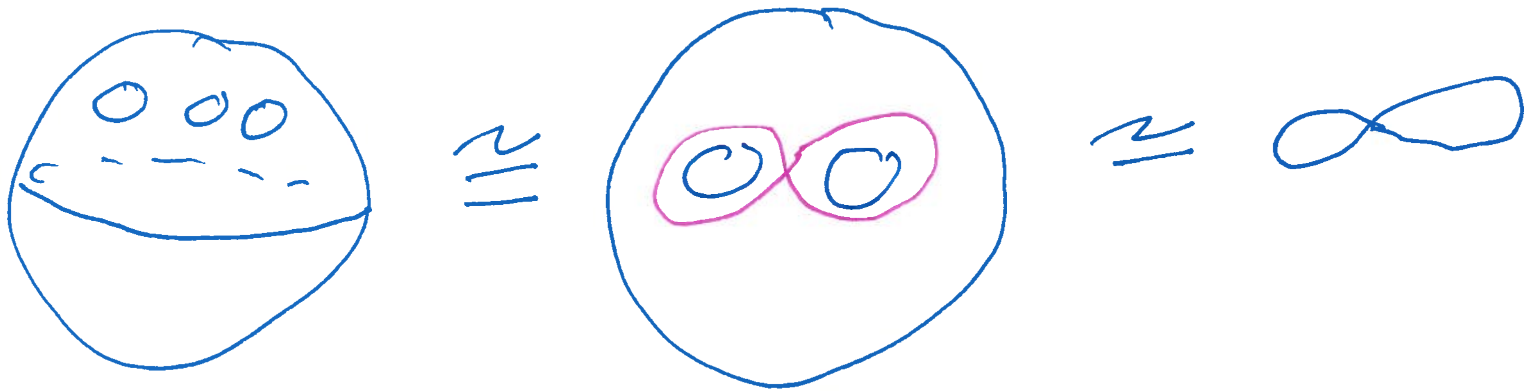
So $(g \circ f_{x_0})_* = g_* \circ (f_{x_0})_*$ is isomorphism.

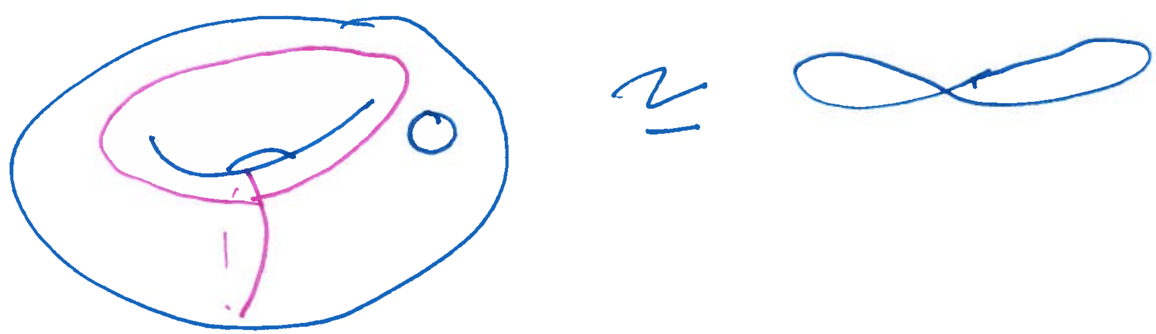
Similary $(f_{x_1} \circ g)_* = (f_{x_1})_* \circ g_*$ is an isom.

$\Rightarrow$ g_* is an isomorphism.

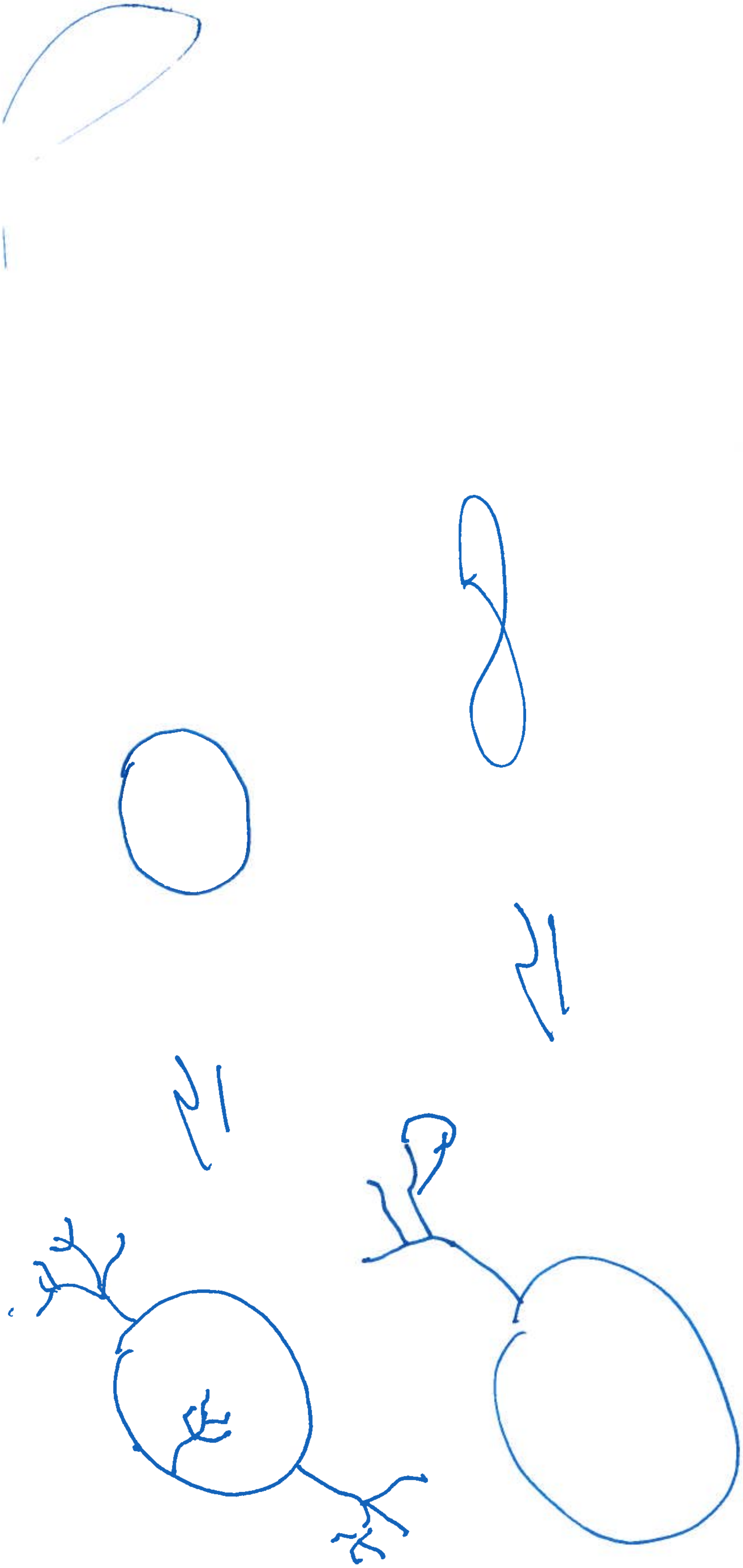
from (1) $(f_{x_0})_* = (g_*)^{-1} \circ \hat{\alpha}$ is an isomorphism. ∎

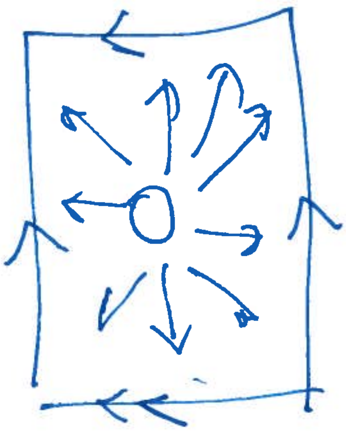
eg

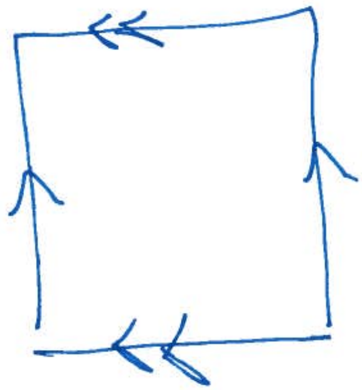


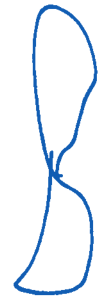


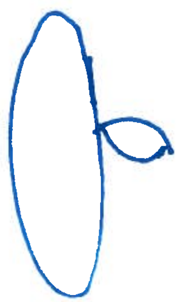
$\pi_1(\infty, x_0) = F_2$ free on 2 symbols.

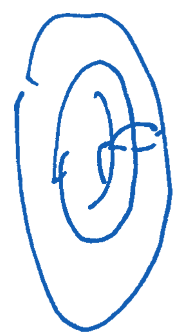


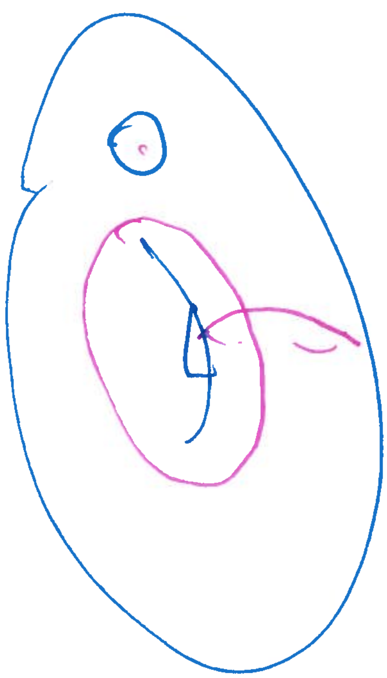












<u>Next Topic</u>: Fundamental group of S^n for $n > 1$ is trivial.

- Seifert-van Kampen expresses $\pi_1(X, x_0)$ in terms of generators and relations.

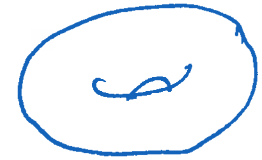
-

DEF: G is a group, $\{g_\lambda\}_{\lambda \in \Lambda}$ is said to generate the group (or are generators) if every $g \in G$ can be written as a finite word (or product) in g_λ and their inverses.

Eg: $\mathbb{Z}$ is gen $\{-1\}$ or $\{1\}$ not by $\{2\}$

$\{2\}$ generates $2\mathbb{Z} \leq \mathbb{Z}$

$\mathbb{Z}^2$ is generated by $\vec{e}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$.



F_2 is generated by say g_1 and g_2.

In $\mathbb{Z}^2$ $e_1 + e_2 = e_2 + e_1$

or $\boxed{e_1 e_2 e_1^{-1} e_2^{-1} = 0}$

$$\mathbb{Z}^2 = \langle e_1, e_2 ; e_1 e_2 e_1^{-1} e_2^{-1} = id \rangle$$

Theorem (part I of S.V.):

$X = U \cup V$ open sets.

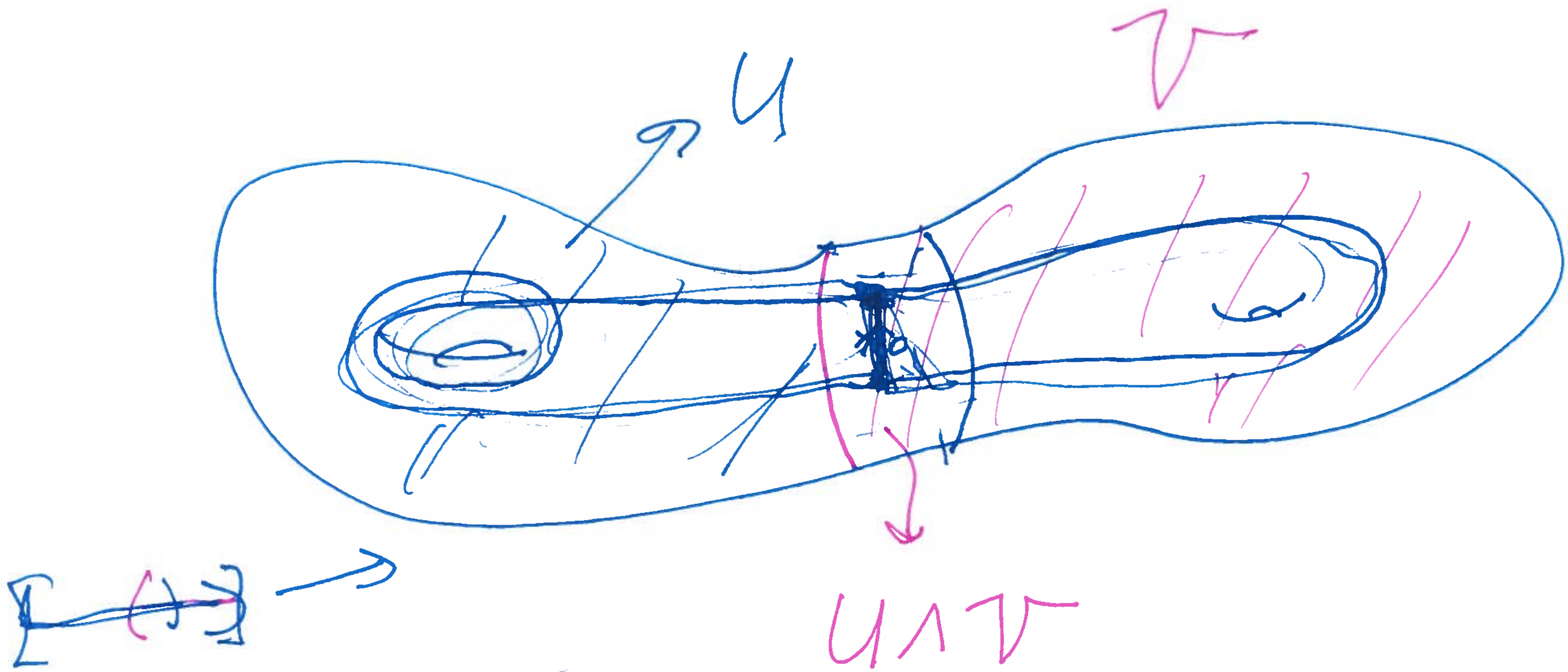
$U \cap V$ is path connected, nonempty, pick $x_0 \in U \cap V$

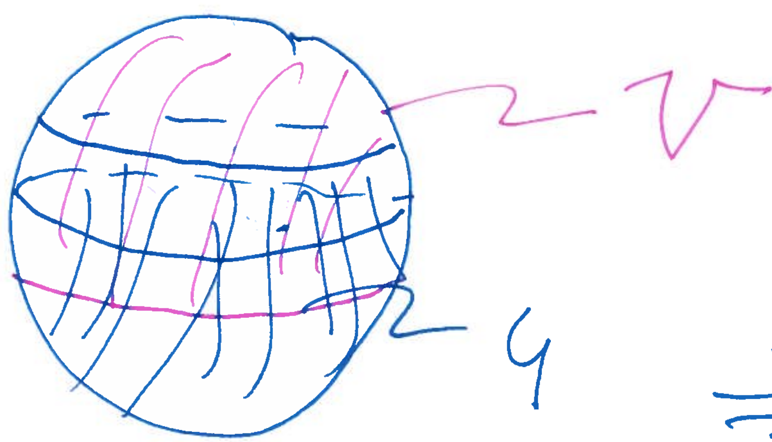
$\Rightarrow$ the images of the maps

$$(\iota_U)_* : \pi_1(U, x_0) \longrightarrow \pi_1(X, x_0)$$

$$(\iota_V)_* : \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$.





$\pi_1(V) = e$

$\pi_1(U) = e$

$\Rightarrow \pi_1(S^2) = e$

Proof $\boxed{\text{Step 0}}$ We need to show that

if f is a loop in X based at x_0

$\Rightarrow$ $f = g_1 * g_2 * g_3 * \cdots * g_n$ where

each g_i is a loop in X based at x_0 and lies entirely in U or V.

Step 1 Goal: subdivide $[0,1]$ as

$0 = a_0 < \cdots < a_n = 1$ so that

(a) $f(a_i) \in U \cap V$

(b) $f([a_{i-1}, a_i]) \subseteq U$ or V

for all i

Proof of the Goal: $f^{-1}(U)$ $f^{-1}(V)$ is a cover of $[0,1]$. Using Lebesgue number Lemma we may find $0 = b_0 < \ldots < b_m = 1$ so that $\forall i$, $f([b_{i-1}, b_i]) \subseteq U$ or V.

Now $f(b_i) \in U \cap V$ $\forall i$ we are done

If not, let i be an index with $f(b_i) \notin U \cap V$. We know $f([b_{i-1}, b_i])$ and $f([b_i, b_{i+1}])$ are in U or V. If $f(b_i) \in U \Rightarrow$ both in U, if $f(b_i) \in V \Rightarrow$ both in V.

In either case, we can delete b_i.
Continue this finite process till the goal is reached.

__Step 2__ Let f_i be a path $(\forall i)$

$$[0,1] \xrightarrow[\text{map}]{\text{affine}} [a_{i-1}, a_i] \xrightarrow{f} X.$$

So $[f] = [f_1] * \cdots * [f_n]$.

For i, choose path α_i in $U \cap V$ from x_0 to $f(a_i)$. Let

$$g_i = \underline{\underline{\alpha_{i-1}}} * f_i * \underline{\underline{\alpha_i^{-1}}}$$

So $[g_1] * \cdots * [g_n] = [f_1] * \cdots * [f_n] = [f]$ yields the desired word in terms of the images. each $[g_i]$ is in either ~~$\pi_1(U)$~~ the image of

$(i_U)_*$ or $(i_V)_*$

Doesn't work on S^1: $U \cap V$ is not path connected