Theorem 1. Let $X$ be a topological space. If $X$ is path connected, then $X$ is also simply connected.

Note: If $X$ is not simply connected, then $X$ is not path connected.

Example: $I = [0, 1]$. In this example, $I$ is simply connected.
Proposition: $S^n, n > 1$, is simply connected.

Main tool: stereographic projection.

$f: S^n \setminus \{ \text{north pole} \} \rightarrow \mathbb{R}^n$
\[ f(x) = \frac{1}{1-x} (x_1, \ldots, x_n) \]

\[ P = (0, \ldots, 1) \quad Z = (0, \ldots, -1) \]

\[ f^n : \mathbb{S}^n - \Sigma \mathcal{C}_3 \to \mathbb{R}^n \]

\[ f^{-1}(\vec{y}) = (\mathcal{I}(\vec{y}) y_1, \ldots, \mathcal{I}(\vec{y}) y_n, 1 - \mathcal{I}(\vec{y})) \]

where \[ \mathcal{I}(\vec{y}) = \frac{2}{1 + \|\vec{y}\|^2} \].

\[ \Rightarrow f \text{ is a homeomorphism} \]
This implies

\[ S^n - \mathbb{S}p^3 \cong \mathbb{R}^n \]

\[ S^n - \mathbb{S}p^3 \cong \mathbb{R}^n - \mathbb{S}o^3 \]

fixes when \( s \in S^2 \)

for all \( \alpha : S^n - \mathbb{S}p^3 \cong \mathbb{R}^n \)

\[ \alpha(x) = -x \]
Proof: Let \( U = S^n - \epsilon \mathbb{R}^3 \)
\[ V = S^n - \epsilon \mathbb{R}^3 \]
\[ V \cup V = S^n - \epsilon \mathbb{R}^3 \cong \mathbb{R}^n - \epsilon \mathbb{R}^3 \]

\( U \cap V \) are homeomorphic to \( \mathbb{R}^n \)

So \( U \cap V \) is path conn.

\[ \Rightarrow \text{ Use Cor.} \]

HW: \( \pi_1(S^1 \times S^1) \cong \mathbb{Z}^2 \) by covering spaces.
Product Theorem:

\[ \Pi^2 = S' \times S' \]

\[ \Pi_1(\Pi^2) = \mathbb{Z} \times \mathbb{Z} \]

\[ P_1: X \times Y \rightarrow X \] projections

\[ P_2: X \times Y \rightarrow Y \]

yields

\[ (P_1)_*: \Pi_1(\Pi^2) \rightarrow \Pi_1(X \times X) \]

\[ (P_2)_*: \Pi_1(\Pi^2) \rightarrow \Pi_1(Y \times Y) \]
Define \( \Phi : \Pi_1(\Sigma x y, (x_0, y_0)) \rightarrow \Pi_1(\Sigma x_0, y, y_0) \)

\[ \Phi(\Sigma x f) = (\prod x f, \iota * \Sigma f) \]

Claim: \( \Phi \) is an isomorphism.

1. \( \Phi \) is surjective

Pick \( (\Sigma g f, \Sigma h f) \) define

\[ (g f) (x) = (g f(x), h f(x)) \]

\[ \Rightarrow \Phi(\Sigma f g) = (\Sigma g f, \Sigma h f) \]
Injective by showing \( \ker \Phi = \emptyset \).
Assume \( \Phi(\xi:\eta) = \xi' \in \emptyset \).

\[ \Rightarrow \quad \forall \eta \in \chi \quad \exists \xi \text{ s.t. } \Phi(\xi) = \eta \text{ by } \chi \xrightarrow{\sim} \chi \text{ say} \]

\[ q \text{ of } \xi' \in \gamma_0 \text{ by } \chi \text{ } \]

Then \( \xi \in \emptyset \) gives \( q = \xi' = 0 \).

\[ \Rightarrow \text{ so } \xi(\xi, \eta) = 0 \text{ in } \Pi_1(\chi \times \gamma, (x_0, y_0)) \text{ is} \]
Computational Tools for Π

(a) covering spaces
(b) products
(c) Seifert-Van Kampen

Another example of (a)

Projective Plane

Define \( x \mapsto x \) on \( S^2 \)

Let \( PP = S^2/\sim \)
$\rho : S^2 \rightarrow S^2 \setminus \mathcal{U}$ is a covering space. (Informally) and $S^2$ is simply connected.

\[ \exists \text{ as sets } \rho \pi_1(\rho^{-1}(x_0)) \cong \rho^{-1}(x_0) \rightrightarrows \rho \pi_1(\rho) \]  
(from $S^1$ computation)

but $\rho^{-1}(x_0)$ is 2 pts. and there is only one group with 2 pts

so $\pi_1(\rho, x_0) = \mathbb{Z}_2$. 
In $\mathbb{R}^3$, a point is a point (passing through zero).

A line is determined by two antipodal points on $S^2$.

$PP$ = all lines in $\mathbb{R}^3$ where each line is a point (passing through origin).
in M glue on a $D^2$ on its single circular boundaries

$\mathcal{T}' / M) = 2 \quad M \cong S^1$

but in $PP = M$ glued to $D^2$

Twice around is the boundary that collapse to a point in the $D^2$