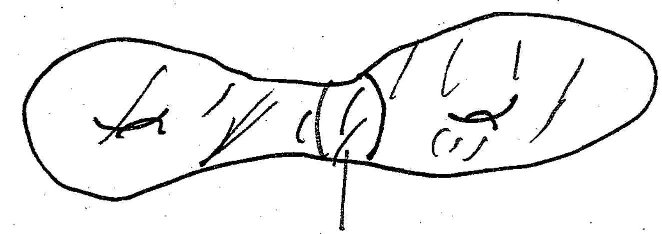
Last time: $X = U \cup V$ open sets

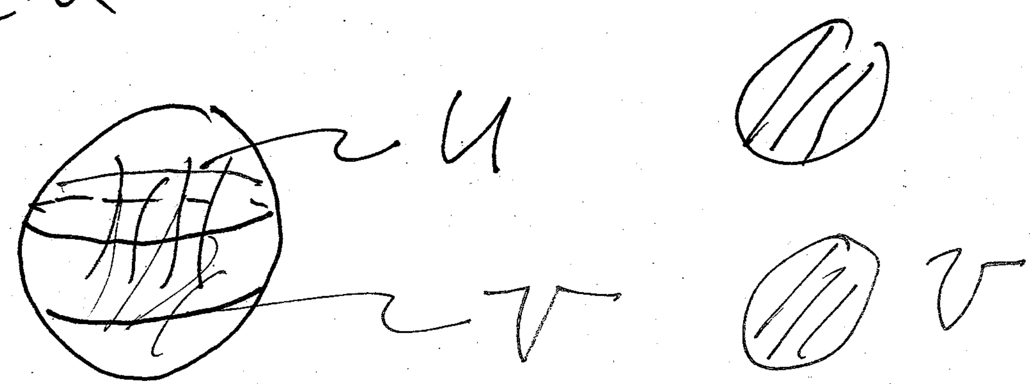
$x_0 \in U \cap V$ which is path connected

$\Rightarrow$ images $(i_U)_*$ and $(i_V)_*$ generate

$\pi_1(X, x_0)$



COR: If further U and V are simply connected $\Rightarrow$ X is also simply conn.



note: $U \cap V$ is not s.c

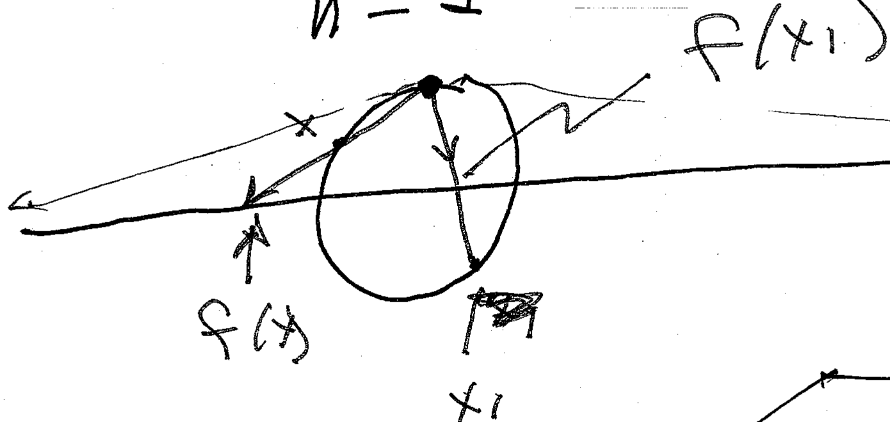
$\pi_1(U \cap V) = \mathbb{Z}$ in this example.

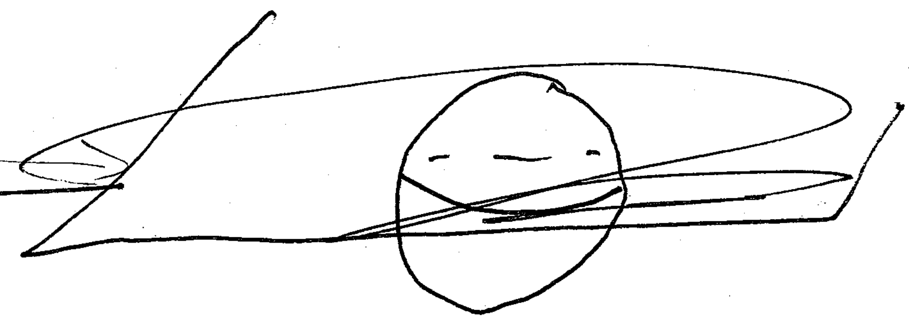
Th'm: S^n $n > 1$ is simply connected.

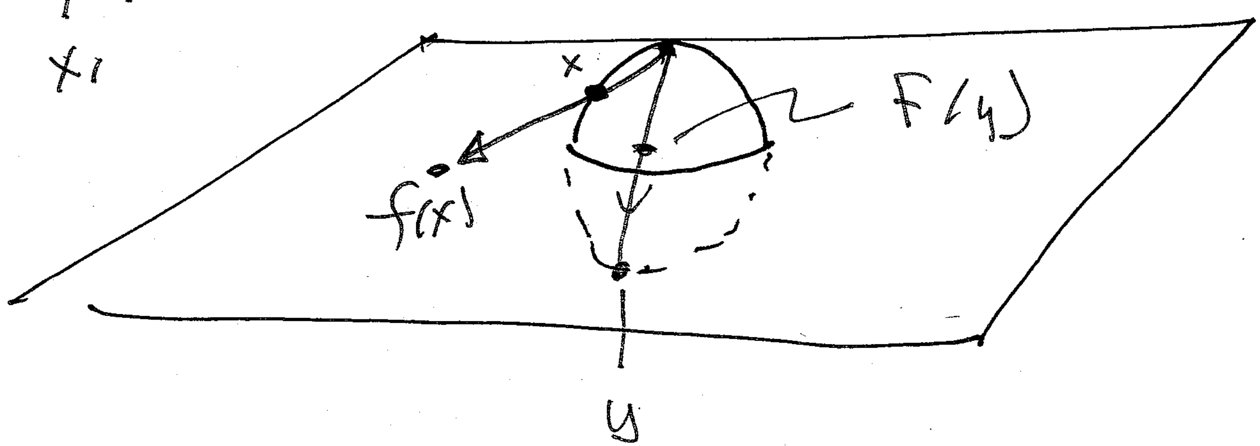
Main Tool: stereographic projection.

$f : S^n - \{\text{north pole}\} \to \mathbb{R}^n$

$n = 1$







$$f(\vec{x}) = \frac{1}{1-x_{n+1}}(x_1, \dots, x_n)$$

$$\underset{\substack{\uparrow \\ \text{north pole}}}{p} = (0, \dots, 1) \qquad \underset{\substack{\uparrow \\ \text{south pole.}}}{q} = (0, \dots, \underset{\substack{\uparrow \\ n+1}}{-1})$$

$$f: S^n - \{p\} \longrightarrow \mathbb{R}^n$$

$$f^{-1}(\vec{y}) = (I(\vec{y})y_1, \dots, I(\vec{y})y_n, 1 - I(\vec{y}))$$

where $I(\vec{y}) = \frac{2}{(1+\|\vec{y}\|^2)}$.

$\Rightarrow f$ is a homeomorphism

This implies

$$S^n - \{p\} \cong \mathbb{R}^n$$

$$S^n - \{p, q\} \cong \mathbb{R}^n - \{\vec{0}\}$$

~~f(-x) f∘S when S(x~~

$$f \circ \alpha : S^n - \{q\} \cong \mathbb{R}^n$$

$$\alpha(x) = -x$$

Proof Let $U = S^n - \{p\}$

$V = S^n - \{q\}$

$U \cap V = S^n - \{p, q\} \cong \mathbb{R}^n - \{\vec{0}\}$

U, V are homeomorphic to $\mathbb{R}^n$

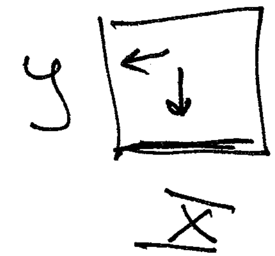
so s.c. $U \cap V$ is path conn.

$\Rightarrow$ use COR.

HW: $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ by covering spaces.

<u>Product Theorem</u>: $\mathbb{T}^2 = S^1 \times S^1$

$\pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$



<u>Th'm</u> $\pi_1(X \times Y, (x_0, y_0))$

$\cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

<u>Proof</u> $p_1 : X \times Y \to X$ projections

$p_2 : X \times Y \to Y$

yields

$(p_1)_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$

$(p_2)_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0)$

Define $\Phi : \pi_1(X \times Y, (x_0, y_0)) \to$

$$\pi_1(X, x_0) \times \pi_1(Y, y_0)$$

$$\Phi([f]) = (p_*[f], q_*[f])$$
$$= ([p \circ f], [q \circ f]).$$

Claim: Φ is an isomorphism.

(1) Φ is surjective

Pick $([g], [h])$ define

$\cancel{[f] =}$ $f(x) = (g(x), h(x))$

$\Rightarrow \Phi([f]) = ([g], [h])$

Injective by showing $\ker \Phi = \{e\}$.

Assume $\Phi([f]) = \{\vec{e}\}$

$\Rightarrow$ $p \circ f \underset{p}{\simeq} e_{x_0}$ by G } say

$q \circ f \underset{p}{\simeq} e_{y_0}$ by H

then $G \times H$ gives $f \simeq e_{(x_0, y_0)}$

so $[f] = e$ in $\pi_1(X \times Y, (x_0, y_0))$ $\blacksquare$

Computational Tools for π_1

(a) covering spaces
(b) products
(c) Seifert-Van Kampen.

Another example of (a)

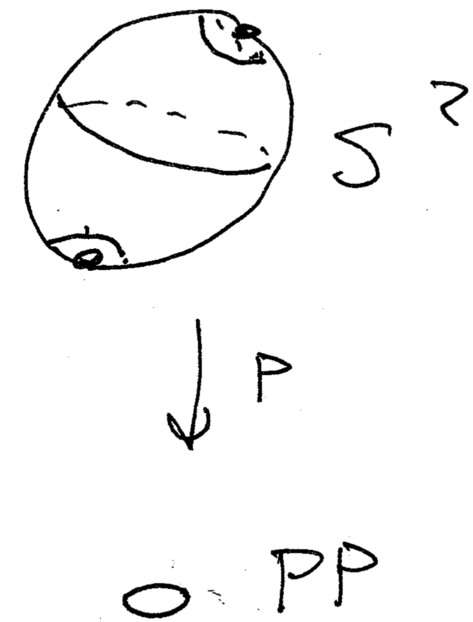
$\boxed{\text{Projective Plane}}$

Define $x \sim -x$ on S^2

Let $PP = S^2/\sim$

$p: S^2 \longrightarrow S^2/\sim \; = PP$ is a covering space. (informally)

and S^2 is simply connecte



$\Rightarrow$ as sets PP

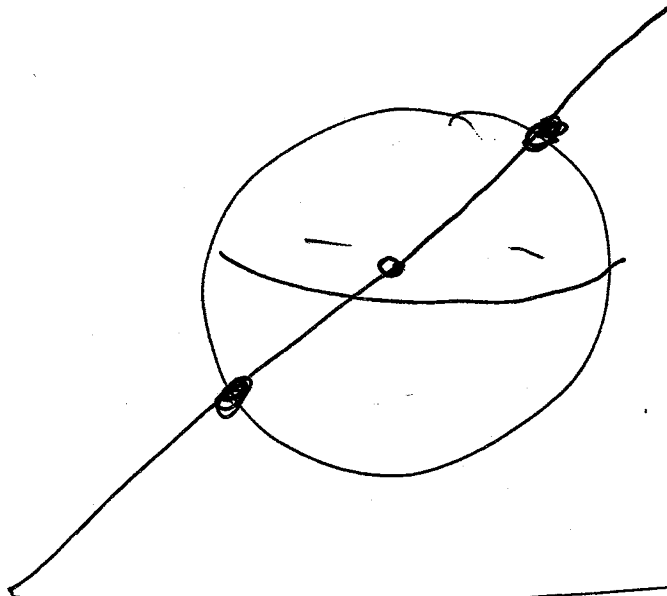
$\pi_1(\quad, x_0) \cong p^{-1}(x_0)$

(from S^1 computation)

but $p^{-1}(x_0)$ is 2 pts. and there is only one group with 2 pts

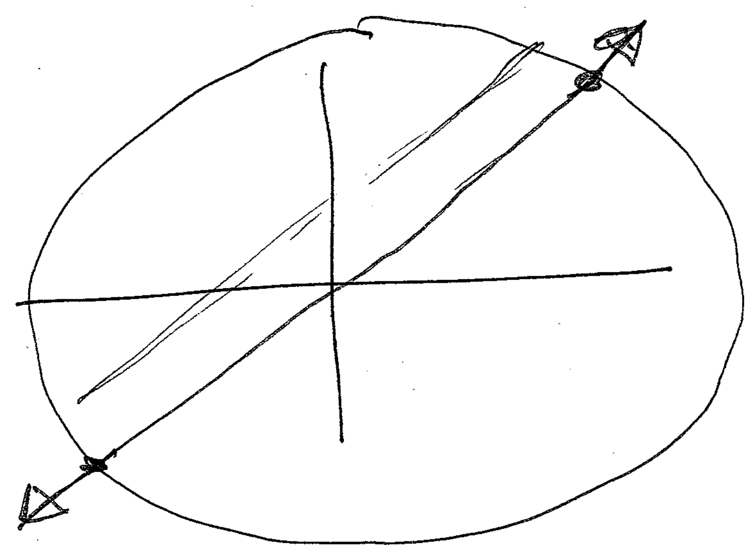
so $\pi(PP, x_0) = \mathbb{Z}_2$.

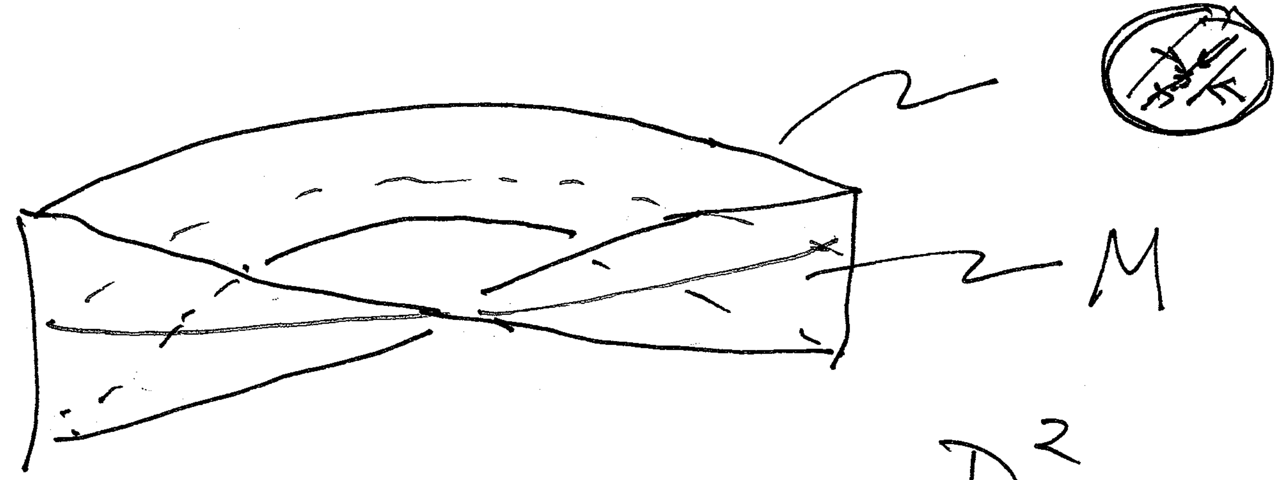
PP = all lines in $\mathbb{R}^3$ where each line is a point (passing through the origins)



a line is determined by two antipodal points on S^2

$\mathbb{R}^2 \cup$ Circle at ∞





in M glue on a D^2 on its single circular boundary

$\pi_1(M) = \mathbb{Z}$ $\quad M \simeq S^1$

but in $PP = M$ glued to D^2

twice around is the boundary that collapse to a point in the D^2