# A little algebra – generators and relations

**DEF:** Given a set $X$, the <u>free group</u> on $X$, $F(X)$, is the collection of all formal finite words in $x_\alpha$ and $x_\alpha^{-1}$ canceling $x_\alpha x_\alpha^{-1}$ to the empty word. The empty word is the identity. $w = x_{\alpha_1} x_{\alpha_2}^{-1} x_{\alpha_3} \cdots$

$$w_1 x_\alpha x_\alpha^{-1} w_2 = w_1 w_2$$

**DEF** The isomorphism class of $F(X)$ just depends on $\mathrm{card}(X)$.

So $F_n$ = free group on n-symbols

$0 < n < \infty$

$F_2 = F(\{a, b\})$ all words

$aba^{-1}bba\ldots$

<u>Lemma</u> Any group $G$ is the homorphic image of a free group <u>or</u> $G \cong F/N$ for $F$ free and $N \triangleleft F$.

normal subgroup.

<u>Proof</u> Let $F = F(G)$

set underlying $G$.

and $\varphi : F \to G$ via

$$\varphi(g_1 g_2 \cdots g_n) = g_1 g_2 \cdots g_n$$

in $F$ in $G$.

obviously a homomorphism

<u>So</u> $N = \ker(\varphi) \Rightarrow$

$$G \cong F/N$$

Eg// $G = \mathbb{Z}^2$ $\quad a = (1,0)$

$b = (0,1)$

$\varphi : F(a,b) \to \mathbb{Z}^2$

$\varphi(aabab^{-1}) \to \cancel{3}a + a + b + a - b$

$\varphi(aba^{-1}b^{-1}) = a + b - a - b = \vec{0}$

So $aba^{-1}b^{-1} \in \ker(\varphi) = N$.

In fact $\ker(\varphi)$ = smallest normal subgroup of $F$ containing $aba^{-1}b^{-1}$

= commutator subgroup.

**DEF** $G$ is called <u>finitely generated</u> if it ~~has~~ is generated by a finite set or equivalently, $G \cong F_n/N$ for $n < \infty$ $N \triangleleft F_n$. ($F_n = F(\text{generators})$)

$G$ is <u>finitely presented</u> if $G \cong F_n/N$ $n < \infty$

$N \triangleleft F_n$ <u>and</u> $N$ is the smallest normal group containing some set $r_1, \ldots, r_k$ called the relations.

page left blank on purpose -- scanning error

In this case $G$ is written

$$G = \langle \underline{g_1, \ldots, g_n} \mid \underline{\underline{r_1, \ldots, r_k}} \rangle$$

or semicolon, colon, ...

---

<u>examples</u> ① $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} = e \rangle$

$\mathbb{Z}^2$ is generated by $a, b$ and

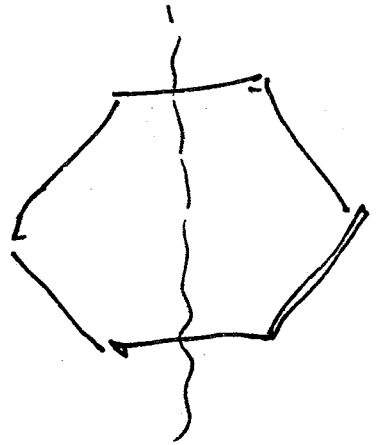
$aba^{-1}b^{-1} = e$ or $ab = ba$

(2) $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \langle a \mid a^n \rangle$

(3) $D_n$ = dihedral group of order $n$

= symmetries of the planar regular polygon with $n$-sides

rotation flip by $2\pi/n$



$$D_n = \langle r, f \mid r^n, f^2, rfrf \rangle$$

$$\boxed{frf = r^{-1}}$$

rotation in a mirror is rotation in opposite direction.

When ~~are~~ do two finite presentations yield isomorphic groups?

$\boxed{\text{Undecidable in general}}$

# OLD FASHIONED Seifert-Van Kampen

$X = U \cup V$ open sets, $U \cap V$ path connected, nonempty <u>and</u> $U, V, U \cap V$ all have finitely presented fundamental groups. and $x_0 \in U \cap V$

$$\pi_1(U, x_0) = \langle a_1 \ldots a_n \mid \alpha_1 \ldots \alpha_m \rangle$$

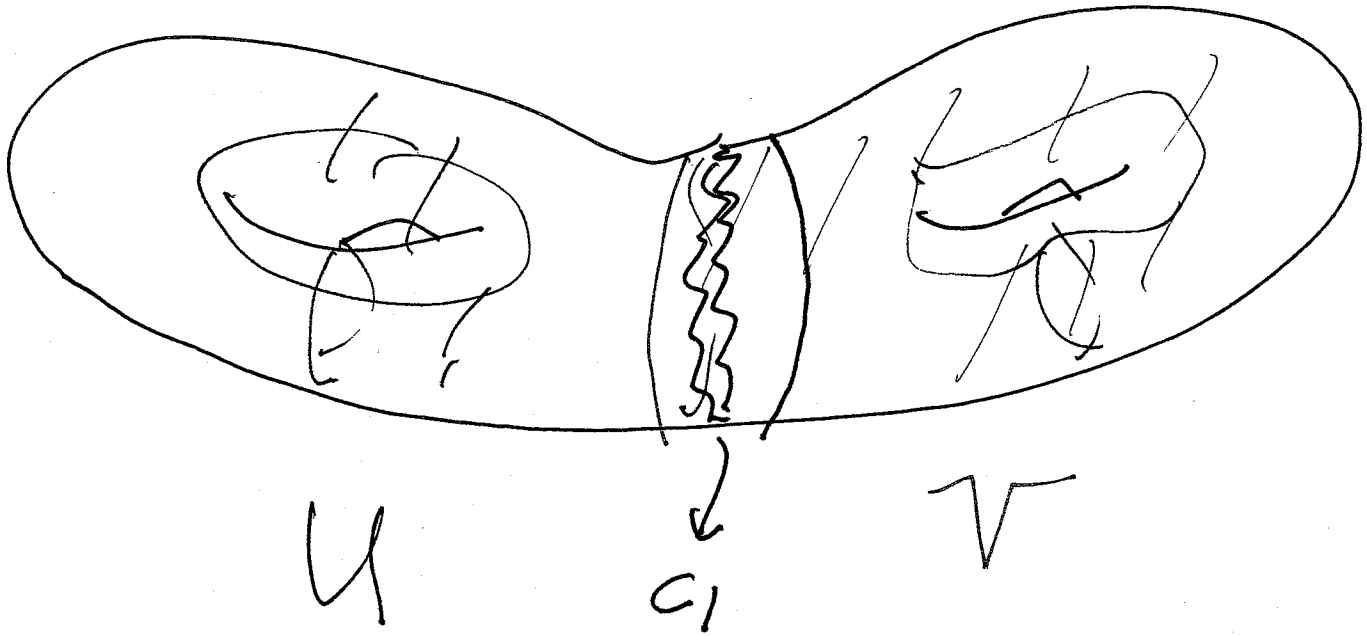
$$\pi_1(V, x_0) = \langle b_1 \ldots b_k \mid \beta_1 \ldots \beta_\ell \rangle$$

$$\pi_1(U \cap V, x_0) = \langle c_1 \ldots c_j \mid \gamma_1 \ldots \gamma_\omega \rangle$$

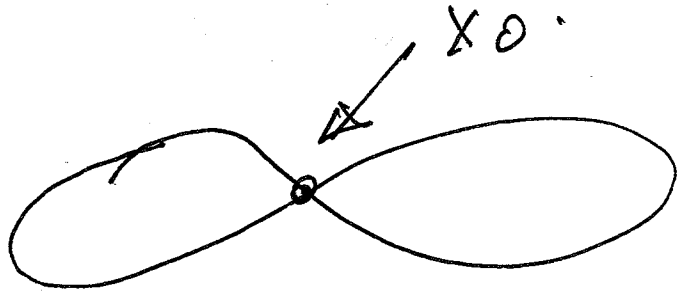
$$\Rightarrow \pi_1(X, x_0) = \langle a_1 \ldots a_n \; b_1 \ldots b_k \mid \alpha_1 \ldots \alpha_m \; \beta_1 \ldots \beta_\ell \; (L_U)_* c_1 ((L_V)_* c_1)^{-1}, \ldots, (L_U)_* c_j ((L_V)_* c_j)^{-1} \rangle$$

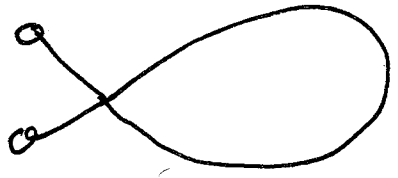
So $X$ has all the generators and relations of the pieces with new relations coming from where $U$ and $V$ join

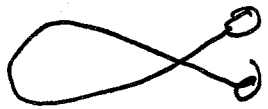
or $(i_U)_* c_i = (i_V)_* c_i$

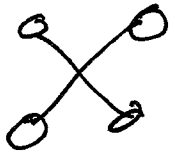


## example




$U =$  $\simeq \bigcirc \quad \pi_1 = \langle a \rangle$

$V =$  $\simeq \bigcirc \quad \pi_1 = \langle b \rangle$

$U \cap V =$  $\simeq \bullet \quad \pi_1 = \{e\}$

deformation retract

$$\pi_1(U, x_0) = \langle a \rangle$$

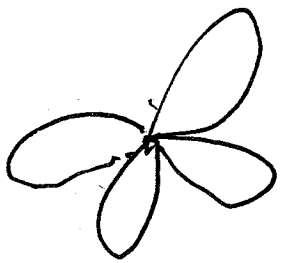
$$\pi_1(V, x_0) = \langle b \rangle$$

$$\pi_1(U \cap V, x_0) = \langle \ \rangle = \langle e \mid e \rangle$$

$$\pi_1(X, x_0) = \langle a, b \mid (L_U)_*(e)\,(L_V)_*(e)^{-1} \rangle$$

$$= \langle a, b \mid e \rangle = \underline{\underline{\langle a, b \rangle}}$$

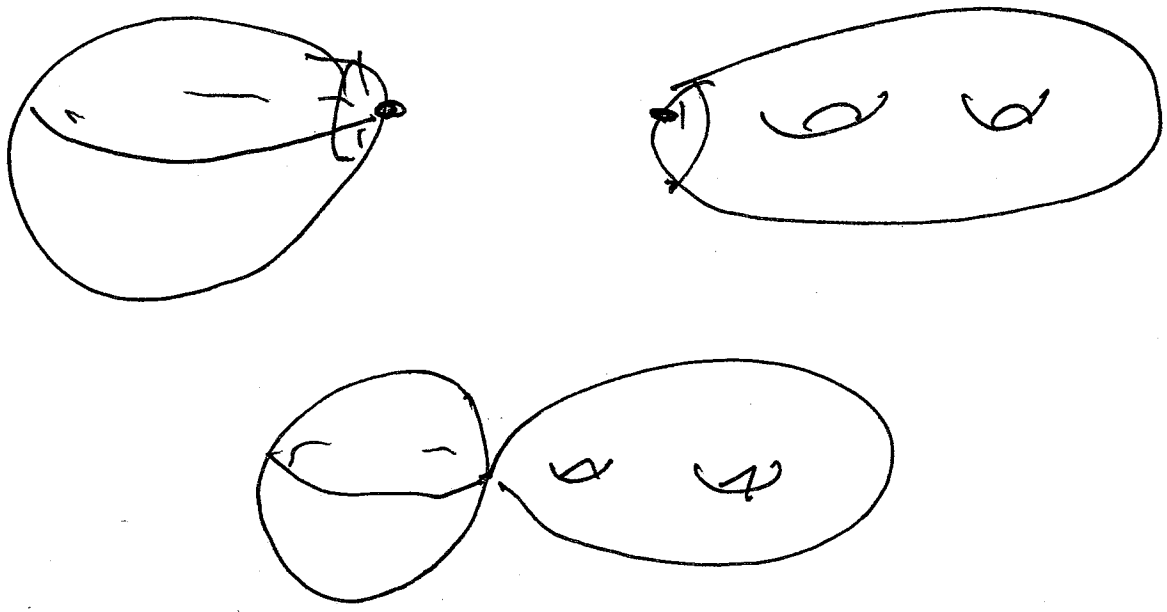
$$= F_2$$



$W_n$ = wedge of $n$-circles

$n < \infty$

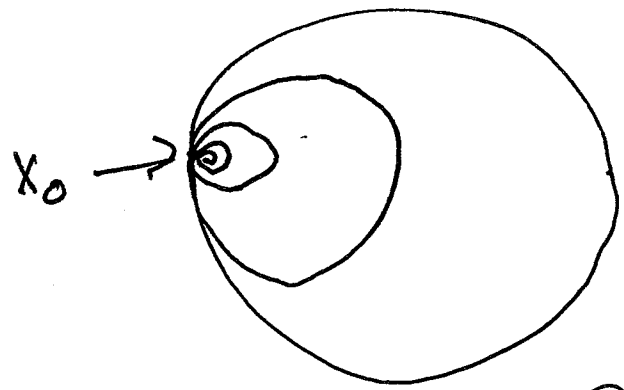
Then $\pi_1(W_n, x_0) \cong F_n$ by induction.



# Infinite (Hawaiian) Earring

$W_n$ = Circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0) \subseteq \mathbb{R}^2$

$$X = \bigcup_{n=1}^{\infty} W_n$$



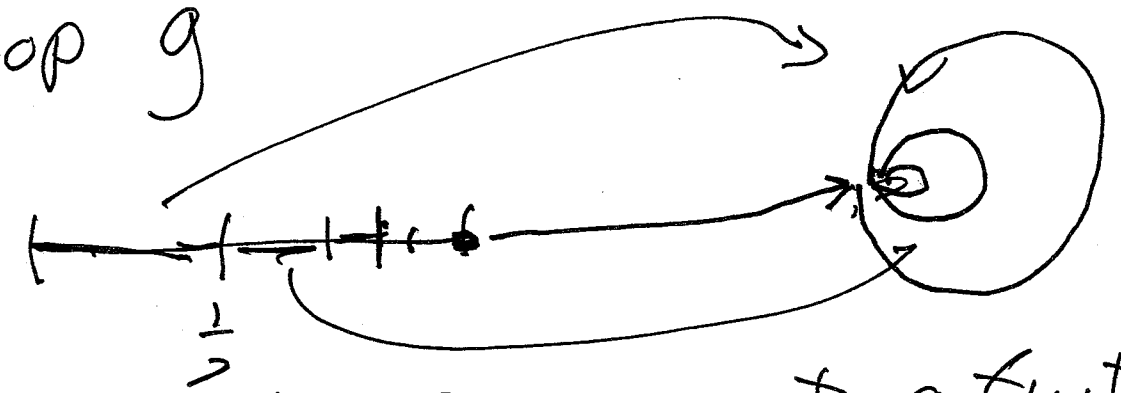

What is $\pi_1(X)$?

Let $f_n$ be the loop determined by $W_n$.

Is $\pi_1$ generated by $\{f_n\}$?

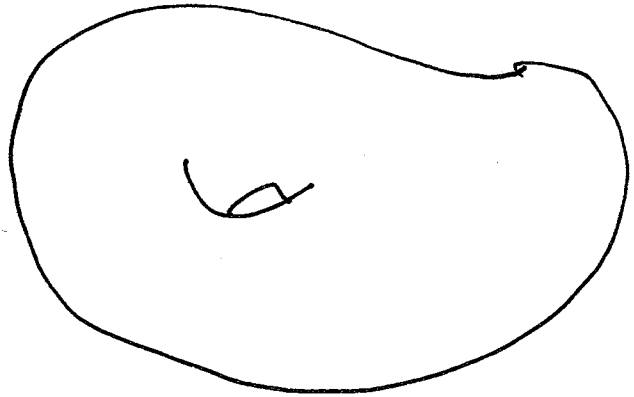
you are only allowed finite words.

Here's a loop $g$




continuous by $[g]$ is not a finite product of $[f_n]$

$\mathbb{T}^2$ again



$\langle a, b \mid \underline{\underline{aba^{-1}b^{-1}}} e \rangle$

$U$

$V$