A little algebra - generators and relations

**DEF:** Given a set \( X \), the free group on \( X \), \( \mathbb{F}(X) \), is the collection of all formal finite words in \( x \) and \( x^{-1} \) canceling \( x \cdot x^{-1} \) to be empty word. \( \emptyset \) the empty word is the identity. \( \mathbb{W} = x_1 x_2^{-1} x_3 \ldots \)

\( w_1 x_2 x_3^{-1} w_2 = w_1 w_2 \)

**DEF** The isomorphism class of \( \mathbb{F}(X) \) just depends on \( \text{card}(X) \).
So $F_n = \text{free group on } n\text{-symbols } \ 0 < n < \infty$

$F_2 = F(\{a, b\})$ all words

$q \tilde{b} \tilde{b} b \tilde{b} q \tilde{b} a q q$

Lemma. Any group $G$ is the homomorphic image of a free group on $\tilde{G} = F/N$ for $F$ free and $N \triangleleft F$. 

$N$ normal subgroup.
Proof: Let $F = F(G)$ set underlying $G$.

and $\Psi: F \rightarrow G$ via

$\Psi(g_1, g_2, \ldots, g_n) = g_1 \cdot g_2 \cdot \ldots \cdot g_n$

Obviously a homomorphism

So $N = \ker(\Psi) \Rightarrow$

$G \cong F/N$
$G = \mathbb{Z}^2 \quad a = (1,0) \\
\quad b = (0,1)$

\[ 
\psi : F(a,b) \to \mathbb{Z}^2 \\
\psi(aa bab^2) \to 3a + b + b + a - b \\
\psi(a b a^{-1} b^{-1}) = a + b - a - b = 0 \\
\] 

So $ab a^{-1} b^{-1} \in \ker(\psi) = N.$

In fact $\ker(\psi) = \text{Smallest normal subgroup of } F\mathbb{Z} \text{ containing } ab a^{-1} b^{-1} \hspace{1cm} = \text{Commutator subgroup.}$
DEF G is called \underline{finitely generated} if it is generated by a finite set or equivalently, \( G \cong \mathbb{F}_n / N \) for \( n < \infty \) \( N \triangleleft \mathbb{F}_n \) (\( \mathbb{F}_n = F(\text{generators}) \))

G is \underline{finitely presented} if

\[ G \cong \mathbb{F}_n / N \quad n < \infty \]

\( N \triangleleft \mathbb{F}_n \) and \( N \) is the smallest normal group containing some set \( r_1, \ldots, r_k \) called the \underline{relations}.
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In this case $G$ is written

$$G = \langle g_1, \ldots, g_n | r_1, \ldots, r_k \rangle$$

or semicolon, colon ...

Examples:

$0 \mathbb{Z}^2 = \langle a, b | a b a^{-1} b^{-1} \rangle$

$\mathbb{Z}^2$ is generated by $a, b$ and

$ab a^{-1} b^{-1} = e$ or $ab = ba$
(2) \[ \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \langle a | a^n \rangle \]

(3) \[ D_n = \text{dihedral group of order } n \]
= Symmetries of the planar regular polygon with n-sides

\[ D_n = \langle r, s | r^n = 1, s^2, rs = sr^{-1} \rangle \]

rotation in a mirror is rotation in opposite direction.
When are two finite presentations yield isomorphic groups?

Undecidable in general
$X = U \cup V$ open sets, $U \cap V$ path connected, nonempty and $U, V, U \cap V$ all have finitely presented fundamental groups, and $x_0 \in U \cap V$

$\pi_1 \left( U, x_0 \right) = \langle a_1, \ldots, a_n | a, \ldots, a_m \rangle$

$\pi_1 \left( V, x_0 \right) = \langle b_1, \ldots, b_k | b, \ldots, b_e \rangle$

$\pi_1 \left( U \cap V, x_0 \right) = \langle c_1, \ldots, c_j | c, \ldots, c_\omega \rangle$

$\Rightarrow \pi_1 \left( \Sigma, x_0 \right) = \langle a_1, \ldots, a_n, b_1, \ldots, b_k |\right.

$\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_e (Lu)_k c_1 (Lu)_k^{-1}, \ldots, (Lu)_k c_j (Lu)_k^{-1} \rangle$
So it has all the generators and relations of the pieces with new relations coming from where \( U \) and \( V \) join.

or \((elu)_r c_i = (lv)_r c_i\)
\[ \Pi_{1}(U, x_0) = \langle a \rangle \]

\[ \Pi_{1}(V, x_0) = \langle b \rangle \]

\[ \Pi_{1}(U \cup V, x_0) = \langle \rangle = \langle a | e \rangle \]

\[ \Pi_{1}(\Sigma, x_0) = \langle a, b | (L_A)_k (e) (L_A)_k (e) \rangle \]

\[ = \langle a, b | e \rangle = \langle a, b \rangle \]

\[ = F_2 \]
\[ W_n = \text{wedge of } n\text{-circles} \quad n \leq \infty \]

Then \[ \pi_1 (W_n) \cong F_n \] by induction.
Infinite (Hawaiian) Earring

\[ W_n = \bigcup_{i=1}^{n} \{ x \in \mathbb{R}^2 : \|x - (i, 0)\| < \frac{1}{n} \} \]

What is \( T_i (\mathbb{F})^2 \)?
Let $F_n$ be the loop determined by $W_n$.

Is $\Pi_1$ generated by $\Sigma F_n$?

You are only allowed finite words.

Here's a loop $g$

Continuous by $\Sigma g$ is not a finite product of $\Sigma F_n$. 
If again
\[ \langle a \mid b^{-1} e b \rangle \Rightarrow 0 \]

\[ 0 \]