Abelian Groups

Last time: $\{G_\alpha\}_{\alpha \in I}$, $G_\alpha \subseteq G$ subgroups

We defined $\bigoplus_{\alpha \in I} G_\alpha$. Called "internal direct sum".

External: Assume G is given with monomorphism $G_\alpha \xhookrightarrow{i_\alpha} G$ $\forall \alpha \in I$

Say G is the external direct sum if

$$G = \bigoplus i_\alpha(G_\alpha)$$

or a little sloppy write

$$G = \bigoplus G_\alpha$$

<u>Lemma</u>: Assume $G = \bigoplus_{\alpha \in I} i_\alpha(G_\alpha) \Rightarrow$

Given any abelian group and homomorphism $h_\alpha : G_\alpha \to H$ for some abelian group H

$\Rightarrow \exists!\, h : G \to H$ with $h \circ i_\alpha = h_\alpha \ \forall \alpha$

$$\begin{tikzcd} G_\alpha \arrow[r, hook, "i_\alpha"] \arrow[dr, "h_\alpha"'] & G \arrow[d, dashed, "\exists!\, h"] \\ & H \end{tikzcd}$$

eg: $\alpha \in \{1, 2\}$

$$\begin{tikzcd}
 & & H & & \\
G_1 \arrow[urr, "h_1"] \arrow[rr, "\iota_1"] & & G \arrow[u, dotted, "h"] & & G_2 \arrow[ull, "h_1"'] \arrow[ll, "\iota_2"']
\end{tikzcd}$$

coproduct in the category of Abelian groups.

Prop.: These properties ~~cate~~ characterize the direct sum. Specifically, Let G' be any abelian with

$L_\alpha' : G_\alpha \hookrightarrow G'$ and $\forall H$ abelian,

family $h_\alpha' : G_\alpha \to H \Rightarrow \exists!\, h'$.

$$\begin{tikzcd} G_\alpha \arrow[r, hook, "L_\alpha'"] \arrow[dr, hook, "h_\alpha'"'] & G' \arrow[d, dashed, "h'"] \\ & H \end{tikzcd}$$

$\Rightarrow$ G is isomorphic to G'. Specifically

$\exists f$ with

$$\begin{tikzcd} G_\alpha \arrow[r, hook, "L_\alpha'"] \arrow[dr, hook, "L_\alpha"'] & G' \arrow[d, "f"] \\ & G \end{tikzcd}$$

f ← isomorphism.

Proof: To start, Treat G' as H and L_α' to be the h_α' or

assume G' has the stated property

$$\begin{tikzcd} G_\alpha \arrow[r, "L_\alpha'"] \arrow[dr, "L_\alpha"'] & G' \arrow[d, dashed, "f'"] \\ & G \end{tikzcd}$$

$\Longrightarrow \exists f'$

switch roles

$$\begin{tikzcd} G_\alpha \arrow[r, "L_\alpha"] \arrow[dr, "L_\alpha'"'] & G \arrow[d, dashed, "f"] \\ & G' \end{tikzcd}$$

Stack the diagrams

$$\begin{tikzcd}
G_\alpha \arrow[r, "L'_\alpha"] \arrow[dr, "L_\alpha"] \arrow[ddr, "L'_\alpha"'] & G' \arrow[d, "f'"] \\
& G \arrow[d, "f"] \\
& G'
\end{tikzcd}
\qquad
\begin{tikzcd}
G_\alpha \arrow[r, "L_\alpha"] \arrow[dr, "L'_\alpha"] \arrow[ddr, "L_\alpha"'] & G \arrow[d, "f"] \\
& G' \arrow[d, "f'"] \\
& G
\end{tikzcd}$$

So
$$\begin{tikzcd} G_\alpha \arrow[r, "i'_\alpha"] \arrow[dr, "i'_\alpha"'] & G' \arrow[d, "f \circ f'"] \\ & G' \end{tikzcd}$$

$$\begin{tikzcd} G_\alpha \arrow[r, "i_\alpha"] \arrow[dr, "i_\alpha"'] & G \arrow[d, "f' \circ f"] \\ & G \end{tikzcd}$$

These also commute with $id_{G'}$ for $f \circ f'$ and id_G for $f' \circ f$. By uniqness $f \circ f' = id_{G'}$ $f' \circ f = id_G \Rightarrow f$ and f' as isom. ∎

Free Abelian Groups

Diagram Definition S is a set. A free Abelian group on S is a group F with a function $\varphi : S \to F$ so that

injective

$$\begin{tikzcd} & F \arrow[dd, dashed, "\exists! f \text{ homomorphism}"] \\ S \arrow[ur, "\varphi"] \arrow[dr, "\psi"'] & \\ & A \end{tikzcd}$$

Any abelian group A any set function ψ.

~~Thus~~ Obs'n This characterizes the Free group wrt to φ and S.

In particular

$$\begin{tikzcd} S \arrow[r, "\varphi'"] \arrow[dr, "\psi"'] & F' \arrow[d, dashed, "\exists ! f'"] \\ & A \end{tikzcd}$$

$\Rightarrow \exists$! isomorphism h with

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & F \arrow[d, "h"] \\ & F' \end{tikzcd}$$

Same proof via stacking

Concrete definition:

Let F be all formal finite sums:

$$\sum_{i=1}^{k} n_i x_{\alpha_i} \qquad n_i \in \mathbb{Z}$$

with all α_i distinct, $x_{\alpha_i} \in S$

with the obvious operation.

Thm: This concrete realization satisfies the diagramatic def.

General Case

Free product

$\{G_\alpha\}_{\alpha \in I}$, $\varphi_\alpha : G_\alpha \to G$ mono.

G is the free product if $\forall H$ (group) and $\psi_\alpha : G_\alpha \to H$ homomorphism $\exists$

$$\begin{tikzcd} & G \arrow[d, "f"] \\ G_\alpha \arrow[ur, "\varphi_\alpha"] \arrow[r, "\psi_\alpha"'] & H \end{tikzcd}$$

This characterizes the free product.

Concrete realization - All words

$$x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_k}$$

adjacent x_{α_L} from different G_α

with obvious operation - concantenation and reduce.

<u>Thm</u>: This satisfies the diagramic def.

Example

generators and relations

$$G_1 = \langle \{a_\alpha\} \mid \cancel{\{b_\beta\}} \{r_\beta\} \rangle$$

$$G_2 = \langle \{b_\alpha\} \mid \{s_\beta\} \rangle$$

$$G_1 * G_2 = \langle \{a_\alpha\} \cup \{b_\alpha\} \mid \{r_\beta\} \cup \{s_\beta\} \rangle$$

free product

$$G_1 = \langle a \mid a^3 \rangle = \mathbb{Z}_3$$

$$G_2 = \langle b \mid b^5 \rangle = \mathbb{Z}_5$$

$$\mathbb{Z}_3 * \mathbb{Z}_5 = \langle a, b \mid a^3, b^5 \rangle$$

$$a^3 = e = b^5$$

different from

$\mathbb{Z}_3 \oplus \mathbb{Z}_5$

<u>NOTATION</u> $G_1 * G_2$

$$\prod^*_{\alpha \in I} G_\alpha \quad \text{or} \quad \mathop{\ast}_{\alpha \in I} G_\alpha$$

<u>Free groups</u> - diagramatic - Same as free Abelian but now $A = H$ is a general group.

<u>Concrete</u> All words $X_{\alpha_1}^{n_1} X_{\alpha_2}^{n_2} \cdots X_{\alpha_K}^{n_K}$

$n_K \in \mathbb{Z}$

concatenation with cancellations

with free group - ~~F and F'~~

$F_1 = \langle a \rangle$

$F_1' = \langle b \rangle$

$F_1 * F_1' = \langle a, b \rangle = F_2$.

$a^{n_1} b^{n_2} a^{n_1} b^{n_2}$