Two versions of the diagram version of ~~M.V.~~ S.V.

Here's a diagram intrinsic to the set up of ~~M.V.~~ SVC

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, hook] & \\
\pi_1(U \cap V) \arrow[ur, hook] \arrow[dr, hook] \arrow[rr, hook] & & \boxed{\pi_1(X)} \\
 & \pi_1(V) \arrow[ur, hook] &
\end{tikzcd}$$

This ~~charte~~ categorizes $\pi_1(X)$

$$\begin{tikzcd}
 & \pi_1(U) \arrow[dr, "\rho_1"] & \\
\pi_1(U \cap V) \arrow[ur] \arrow[dr] \arrow[rr, "\rho_3"] & & H \\
 & \pi_1(V) \arrow[ur, "\rho_2"'] &
\end{tikzcd}$$

H group

Given ρ's and $H \Rightarrow \exists!\ \Gamma : \pi(X) \to H$

with

$$\begin{tikzcd}
 & \pi_1(X) \arrow[d, "\Gamma"] \\
\pi_1(U) \arrow[ur] \arrow[r, "\rho_1"'] & H
\end{tikzcd}
\quad
\begin{tikzcd}
 & \pi_1(X) \arrow[d, "\Gamma"] \\
\pi_1(V) \arrow[ur] \arrow[r, "\rho_2"'] & H
\end{tikzcd}
\quad
\begin{tikzcd}
 & \pi_1(X) \arrow[d, "\Gamma"] \\
\pi_1(U \cap V) \arrow[ur] \arrow[r, "\rho_3"'] & H
\end{tikzcd}$$

This characterizes $\pi_1(X)$

Version 2: $\quad X = U \cup V \quad U, V$ open.

$U, V, U \cap V$ all nonempty + path conn.

$x_0 \in U \cap V$. H is any group.

$\phi_1 : \pi_1(U, x_0) \to H$ are given

$\phi_2 : \pi_1(V, x_0) \to H \quad \Longrightarrow \quad \exists!\, \Phi$ with

$$\cancel{\pi_1(U \cap V, x_0) \hookrightarrow \pi_1(U, x_0)}$$

$$
\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[d, "j_1"] \arrow[dr, "\phi_1"] & \\
\pi_1(U \cap V, x_0) \arrow[ur, "\iota_1"] \arrow[r] \arrow[dr, "\iota_2"'] & \pi_1(X, x_0) \arrow[r, dashed, "\Phi"] & H \\
 & \pi_1(V, x_0) \arrow[u, "j_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}
$$

$\Rightarrow \exists! \, \Phi$ with $\iota_1, \iota_2, j_1, j_2$

inclusions

$((\iota_U)_* = j_1$

Proof: Uniqueness - i.e. if Φ exists $\Rightarrow$ it is unique.

~~Φ~~ We previously proved the images of j_1 and j_2 together generate $\pi_1(X, x_0)$.

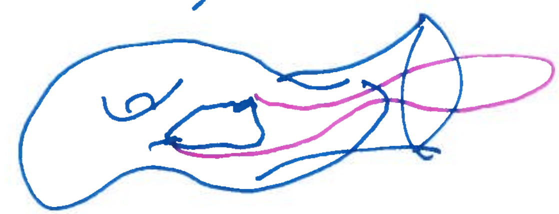
$$\Phi(j_1(\alpha)) = \phi_1(\alpha) \qquad \alpha \in \pi_1(U, x_0)$$

$$\Phi(j_2(\alpha)) = \phi_2(\alpha) \qquad \alpha \in \pi_1(V, x_0)$$

now $j_1(\alpha)$ and $j_2(\alpha)$ are generators and the action of Φ on them is determined.

$\Rightarrow$ Φ on generators is determined so Φ is determined.

__Existence:__ Notation: Assume f is a path in X. If $f \subseteq U$ [sloppy notation for $\mathrm{image}(f) \subseteq U$] write $[f]_U$ for the path homotopy class of f in U.



Similarly for V and $U \cap V$.

We produce Φ by successively defining it for larger classes of inputs.

Step 1 Assume f is a loop based at x_0 and $f \subseteq U$ or $f \subseteq V$.

if ~~Im f~~ $\mathrm{Im}(f) \subseteq U$ let $\rho(f) = \phi_1([f]_U)$

$\mathrm{Im}(f) \subseteq V$ let $\rho(f) = \phi_2([f]_V)$

if $\mathrm{Im}(f) \subseteq U \cap V$

$\Rightarrow \phi_1([f]_U) = \phi_1 i_1([f]_{U \cap V})$

$\phi_2([f]_V) = \phi_2 i_2([f]_{U \cap V})$

Since $\phi_1 i_1 = \phi_2 i_2$ by hypothesis

So ρ is well defined.

Note two properties of ρ

(1.1) $[f]_U = [g]_U$ or $[f]_V = [g]_V$

$$\Rightarrow \rho(f) = \rho(g)$$

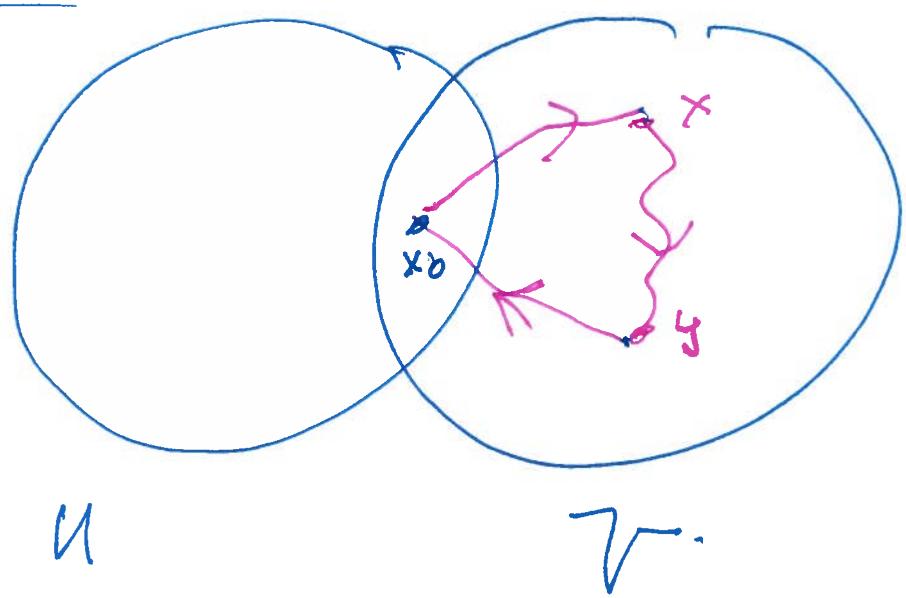
(1.2) If $f, g \subseteq U$ or $f, g \subseteq V$ ~~and $f * g$ is~~ $\Rightarrow$

$$\rho(f * g) = \underbrace{\rho(f) \cdot \rho(g)}_{\text{in } \underline{H}}$$

Since ϕ_1 and ϕ_2 are homomorphisms.

__Step 2__ Extend ρ to $\tilde{\Gamma}$ defined on __paths__ which are in U or V.

picture.



$f \subseteq V$

We construct path α_x connes x_0 to x as follows. Using the fact that $U, V, U \cap V$ are p.c.

$$x = x_0 \Rightarrow \alpha_x \equiv x_0$$

$$x \in U \cap V \Rightarrow \alpha_x \subseteq U \cap V$$

~~$x \in U$~~

$$x \in U - V \Rightarrow \alpha_x \subseteq U$$

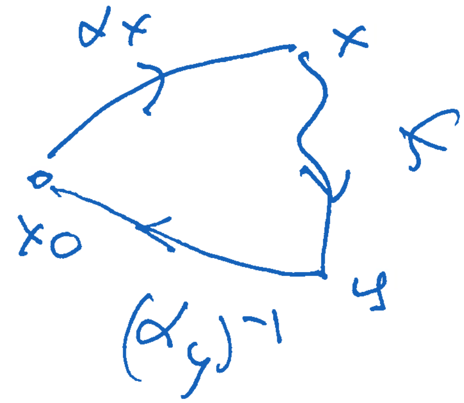
$$x \in V - U \Rightarrow \alpha_x \subseteq V$$

If $f \subseteq U$ or $f \subseteq V \Rightarrow$ define the loop $L(f) = \alpha_x * f * (\alpha_y)^{-1}$ where $x = f(0)$ $y = f(1)$.

Notice by construction

$L(f) \subseteq U$ if $f \subseteq U$

and $L(f) \subseteq V$ if $f \subseteq V$



So ρ is defined on $L(f)$

and define $\tau(f) = \rho(L(f))$.

Properties

2.0: τ extends ρ since when

f is a loop, $f \subseteq U$ or $f \subseteq V$

$\Rightarrow L(f) = e_{x_0} * f * e_{x_0}$

$\Rightarrow [L(f)] = [f]$

(2.1)! If $[f]_U = [g]_U$ or $[f]_V = [g]_V$

$\Rightarrow \sigma(f) = \sigma(g)$.

Proof! If $[f]_U = [g_U]$

$\Rightarrow f \underset{p}{\simeq} g$ in U (they start and end at same place)

$$L(f) = \alpha_x * f * \alpha_y^{-1}$$
$$L(g) = \alpha_x * g * \alpha_y^{-1}$$

$\Rightarrow [L(f)] = [L(g)]$

$\Rightarrow \rho(L(f)) = \rho(L(g))$ (by 1.1)

$\Rightarrow \sigma(f) = \sigma(g)$.

$\boxed{2.2}$ $f, g \subseteq U$ and $f, g \subseteq V$

and $f * g$ is defined $\Rightarrow$

$$\nabla(f * g) = \cancel{I(f) \cdot I(g)}$$
$$\nabla(f) \cdot \nabla(g)$$

Since $f(1) = g(0)$

$$L(f) * L(g) = \left(\alpha_x * f * \alpha_y^{-1}\right) * \left(\alpha_y * g * \alpha_z^{-1}\right)$$

with $z = g(1)$

$$\Rightarrow [L(f) * L(g)] = [\alpha_x * f * g * \alpha_y^{-1}]$$

Take ρ to get conclusion $= [L(f * g)]$