Classical versions of SVK.

<u>**Th'm**</u> Usual hypotheses on $U, V, U \cap V, x_0$

Define $j : \pi(U) * \pi(V) \to \pi(X)$

$[\ \pi(U) = \pi_1(U, x_0)\ ]$ $U \ni a$

$j(g_1 g_2 g_3 \ldots) = j_1(g_1)\, j_2(g_2) \ldots$

$g_{odd} \in \pi(U)$

$g_{even} \in \pi(V)$

$j_1 = (i_U)_*$ $j_2 = (i_V)_*$.

$\Rightarrow$ j is surjective.

and $N = \ker j$ is the smallest normal subgroup that contains all

$$(L_1(g))(L_2(g))^{-1} \quad \forall g \in \pi(X)$$

L_1: $\pi(U \cap V) \hookrightarrow \pi(U)$

L_2: $\pi(U \cap V) \hookrightarrow \pi(V)$

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maps on a diagram

$$\begin{tikzcd}
 & \pi(U) \arrow[dr, "j_1"] & \\
\pi(U \cap V) \arrow[ur, "L_1"] \arrow[rr, "L_*"] \arrow[dr, "L_2"'] & & \pi(X) \\
 & \pi(V) \arrow[ur, "j_2"'] &
\end{tikzcd}$$

all induced by inclusion.

Before the proof - ~~gener~~ presentation

$$\pi(U) = \langle \{a\} \mid \{\alpha\} \rangle$$
$$\pi(V) = \langle \{b\} \mid \{\beta\} \rangle$$
$$\pi(U \cap V) = \langle \{c\} \mid \{\gamma\} \rangle$$

$$\pi(X) = \langle \{a\}, \{b\} \mid \{\alpha\}, \{\beta\}, \{ i_1(c)(i_2(c))^{-1} \} \rangle$$

Proof Step 1: We proved that the images of j_1 and j_2 generate $\pi(X)$. This implies j is surjective.

Step 2: We show $N \subseteq \ker j$. Since $\ker j$ is normal it suffices to show $L_1(g) L_2(g)^{-1} \in \ker j$ $\forall g \in \pi(U \cap V)$.

all

commutivity.

$$j L_1(g) = j_1 L_1(g) = i_*(g) = j_2 L_2(g) = j L_2(g)$$

$i : U \cap V \hookrightarrow V$.

$$\Rightarrow j(L_1(g) L_2(g)^{-1}) = e$$

Step 3 We show $N = \ker j$

$N \subseteq \ker(j)$ and j is an epimorphism

j induces ~~and~~ an epimorphism

$k : (\pi_1(U) * \pi_1(V))/N \rightarrow \pi(X)$

and $N = \ker(j) \Leftrightarrow k$ is an isomorphism

So the task now is to show k is injective. We do this by finding Φ a left inverse to k.

i.e. $\Phi k = \mathrm{id} \Rightarrow k$ is inject.

We construct Φ ~~from~~ de diagramic sur.

$$\begin{tikzcd}
 & \pi(U) \arrow[d, "J_1"] \arrow[dr, "\phi_1"] & \\
\pi(U \wedge V) \arrow[ur, "L_1"] \arrow[r, "L_*"] \arrow[dr, "L_2"'] & \pi(X) \arrow[r, dashed, "\Phi"] & H \\
 & \pi(V) \arrow[u, "J_2"'] \arrow[ur, "\phi_2"'] &
\end{tikzcd}$$

Let $H = \left(\pi(U) * \pi(V)\right)/N$.

$\phi_1 : \pi(U) \longrightarrow H$ via

take $\pi_1(U)$ into the first factor of

$\pi(U) * \pi(V) \Rightarrow$ project to

$$H = \left(\pi(U) * \pi(V)\right)/N$$

Similarly $\phi_2 : \pi(V) \longrightarrow H$.

we need $\phi_1 i_1 = \phi_2 i_2$

If $g \in \pi(U \cap V)$

$\Rightarrow$ $\phi_1 i_1(g)$ is the coset $(i_1 g)N$

$\phi_2 i_2(g)$ is the coset $(i_2 g)N$

but $(i_1 g)(i_2 g^{-1}) \in N$

$\Rightarrow$ $\phi_1 i_1(g) = \phi_2 i_2(g)$

so diagram commutes.

So $\exists\ \Phi : \pi(X) \to H$

claim $\Phi k = \mathrm{Id}$

OR $\Phi \circ k(gN) = gN$

when $g \in \pi_1(Y)$ a $\pi(V)$

But $g \in \pi(Y) \Rightarrow$

$k(gN) = j(g) = j_1(g)$.

$\Rightarrow \Phi k(gN)) = \Phi j_1(g) = \phi_1(g) = gN$

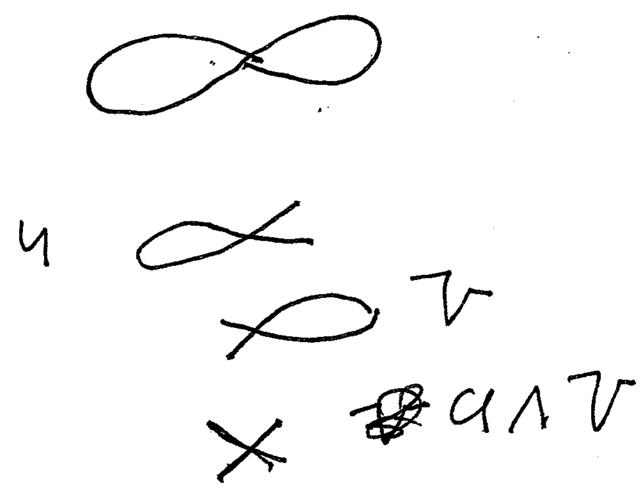
Similar if $g \in \pi(V)$ $\blacksquare$

<u>COR</u> Hypotheses of SVK

and $U \cap V$ is simply connected

$N = \langle e \rangle$

so $\pi(X) \cong \pi(U) * \pi(V)$



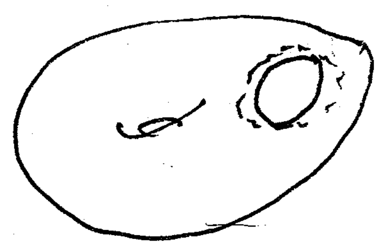
$$\Rightarrow \pi(X) = \underbrace{\mathbb{Z}}_{F(1)} * \mathbb{Z} = F(2)$$

<u>CORR</u> If V is S.C $\Rightarrow$

$$\pi(X) \cong \pi(U)/N$$

where N is the least normal subgroup containing the image of $\pi(U \cap V) \longrightarrow \pi(U)$

Typically V is a disk



Amalgamated Free Product

Given ~~gr~~ groups G_1 and G_2 another group F and monomorphism

$$\varphi_1 : F \to G_1 \qquad \varphi_2 : F \to G_2$$

then the amalgamated free product with this data is

$$G_1 *_F G_2 = (G_1 * G_2)/N$$

N is the smallest normal ~~g~~ subgroup containing all $(\varphi_1(f))(\varphi_2(f))^{-1}$

[So F has isomorphic copies in G_1 and G_2 and "glue together" along] diagram

$$
\begin{tikzcd}
 & G_1 \arrow[d] \arrow[dr, "\phi_1"] & \\
F \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & G_1 *_F G_2 \arrow[r, dashed, "\exists!\Phi"] & H \\
 & G_2 \arrow[u] \arrow[ur, "\phi_1"'] &
\end{tikzcd}
$$

This characterizes the amalgamated free product

and here is a version using presentations.