Top 2.1

On exams asked to state SVK -
suggest you use Thm 70.2 version with
full hypothesis

Building spaces from existing spaces

Quotient Topology: X is a top space and Y is just a set. $f: X \to Y$ surjective.
The quotient topology on Y, $\mathcal{T}$, is defined by V open in $Y \Leftrightarrow f^{-1}(V)$ is open in X.

Then $f: X \to (Y, \mathcal{T})$ is continuous.

Assume $f: X \to (Y, \mathcal{T}')$ is also cont.

Then V' open in $\mathcal{T}'$ $\Rightarrow$ $f^{-1}(V')$ open in X. Thus V' is open in $\mathcal{T}$

So $\mathcal{T}' \subseteq \mathcal{T}$

Thus the quotient topology is the largest topology that makes f continuous.

Two common applications

(1) $\sim$ is equiv on X with classes denoted $[x]$. Let Y be the set of equiv. classes let $f: X \to Y$ be $x \mapsto [x]$. and give Y the quotient topology. Sometimes written $X/\sim$

eg// $X = \mathbb{R}$, $x \sim y \iff x - y \in \mathbb{Z}$

$\Rightarrow X/\sim \;\cong S^1$ $(\mathbb{R}/\mathbb{Z})$

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(2) $X = \coprod_{x \in \Lambda} A_x$ ($\coprod$ = "disjoint union")

called a decomposition of X

Define $f: X \to \Lambda$ ~~and define~~

~~f:~~ via $f(x)$ is the unique x with $x \in A_x$. and give Λ the quotient topology.

eg/ $X = [0,1]$, $X = \coprod_{x \in (0,1)} \{x\} \coprod \{0,1\}$

get circle again.

(1) and (2) are the same

~~let~~

Given $\sim$ $\Rightarrow$ $A_\lambda = [x]$

Given decom $\Rightarrow$ $x \sim y \Leftrightarrow$ $x, y \in A_\lambda$

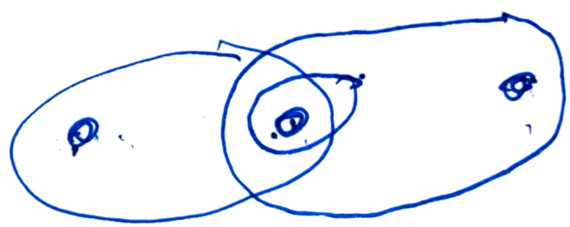
<u>Thm</u>: If X is cpt, conn or path con

$\Rightarrow$ so is $X/\sim$ since f is cont.

Caution: If X is Hausdorff $\Rightarrow$ $X/\sim$ doesn't have to be.

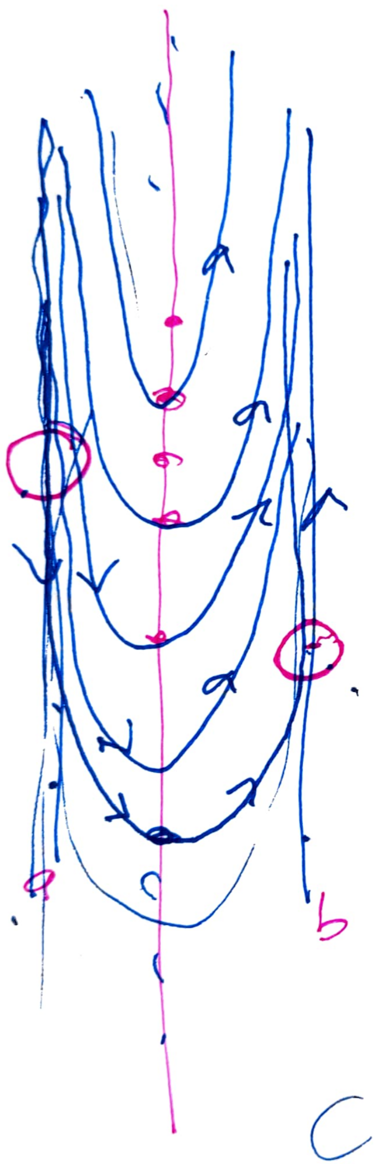
$\boxed{\text{eg//}}$ $[0,1] = \{0\} \amalg (0,1) \amalg \{1\}$

$X/\sim$ is three points



NOT HD.

More sophisticated example – Reeb components



as a set $X/\sim$

$\{a\} \amalg \mathbb{R} \amalg \{b\}$

but $\{a\}$ and $\{b\}$ are not separated by disjoint open sets

Identification (lots of other names) Spaces

"Gluing portions of spaces together"

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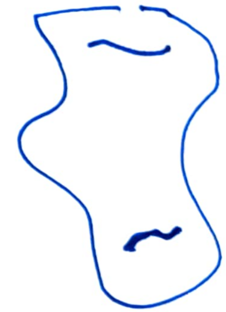
(1) A and $B \subseteq X$ disjoint. A homeomorph $h: A \to B$ is given. Define an equiv rel on X

$$
\begin{aligned}
a &\sim h(a) & a &\in A \\
b &\sim h^{-1}(b) & b &\in B \\
x &\sim x & x &\notin A \cup B
\end{aligned}
$$



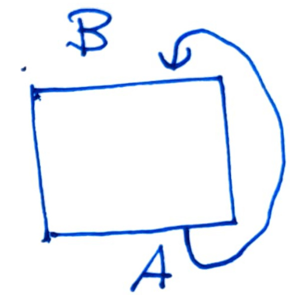
$X/\sim$ is sometimes written X/h.

eg//

$\underline{X} = [0,1].$

$A = [0,1] \times \{0\}$

$B = [0,1] \times \{1\}$



$h : A \to B \qquad h(x,0) = (x,1)$

$\underline{X}/h$

cylinder

or $S^1 \times [0,1]$

(2) $A_1, \dots, A_n$, $B_1, \dots, B_n$

$h_i : A_i \to B_i$ homeomorphism $\forall i$

$a \sim h_i(a)$ $a \in A_i$

$b \sim h_i^{-1}(b)$ $b \in B_i$

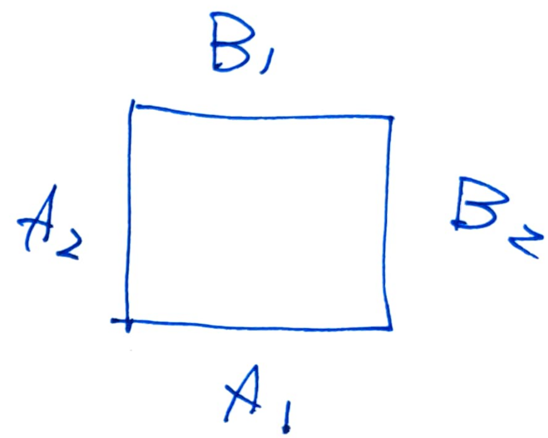
$x \sim x$ $x \notin \bigcup A_i \cup \bigcup B_i$

__Note__ $A_i \cap B_j$ is allowed as long as the h's match.

$h_i(x) = h_j^{-1}(x)$ $x \in A_i \cap B_j$

Example

$X = [0,1] \times [0,1]$



$A_1 = [0,1] \times \{0\}$

$A_2 = \{0\} \times [0,1]$

$B_1 = [0,1] \times \{1\}$

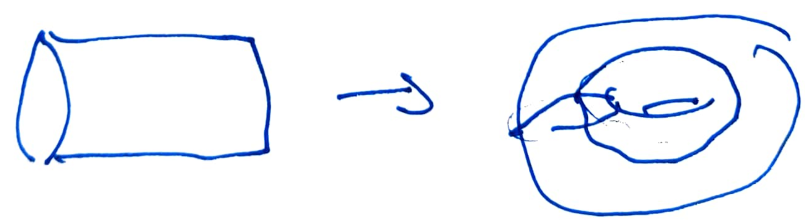
$B_2 = \{1\} \times [0,1]$

$h_1 : A_1 \to B_1 \qquad (x,0) \mapsto (x,1)$

$h_2 : A_2 \to B_2 \qquad (0,x) \mapsto (1,x)$

NOTICE matching at corners

$X/\sim$



…nt to a point

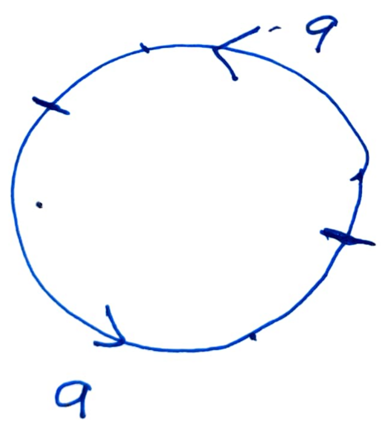
3 ~~so~~ You can have $A = B$

and use $h: A \to A$ a homeomorphism.

$X = D^2 \quad A = S^1$

$h: A \to A \quad h(x) = -x$

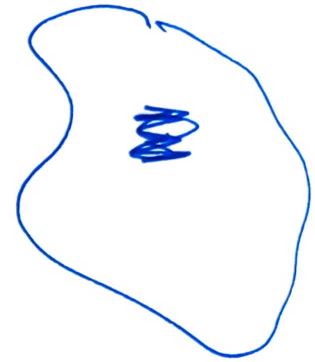
or this example can be split



③ Collapse a subset to a point

$A \subseteq X$ define a decomposition

$$A \amalg \bigcup_{x \notin A} \{x\}$$



collapses A to a point –

– written X/A

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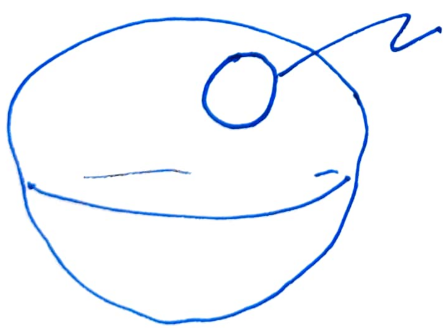
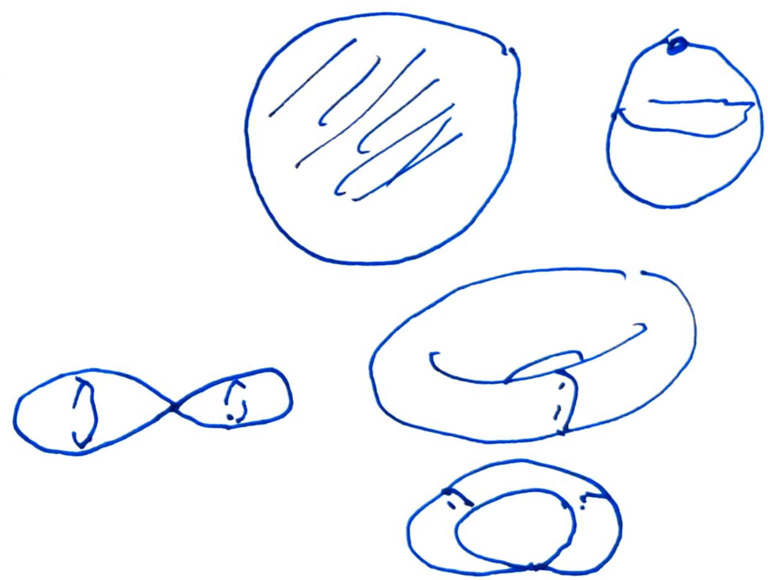
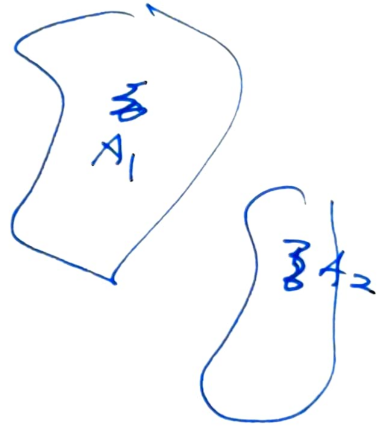
eg: $D^2/S^1 \cong S^2$



$A =$



"Sums" of spaces. - glue multiple spaces togethers.



$A_1 \subseteq X_1$

$A_2 \subseteq X_2$

$h: A_1 \to A_2$ homeomorphism.

Define $\sim$ on $A_1 \sqcup A_2$

$a \sim h(a)$ $a \in A_1$

$b \sim h^{-1}(b)$ $b \in A_2$

$x \sim x$ $\cancel{x \notin A_1 \cup A_2}$ $\forall x$

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get quotient topology

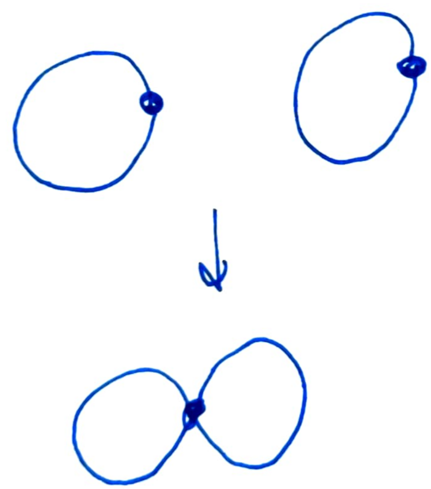
X_1 A_2

Example $a_0 \in X_1$ $\quad h(a_0) = b_0$

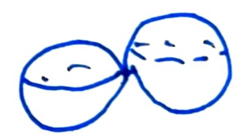
$b_0 \in X_2$

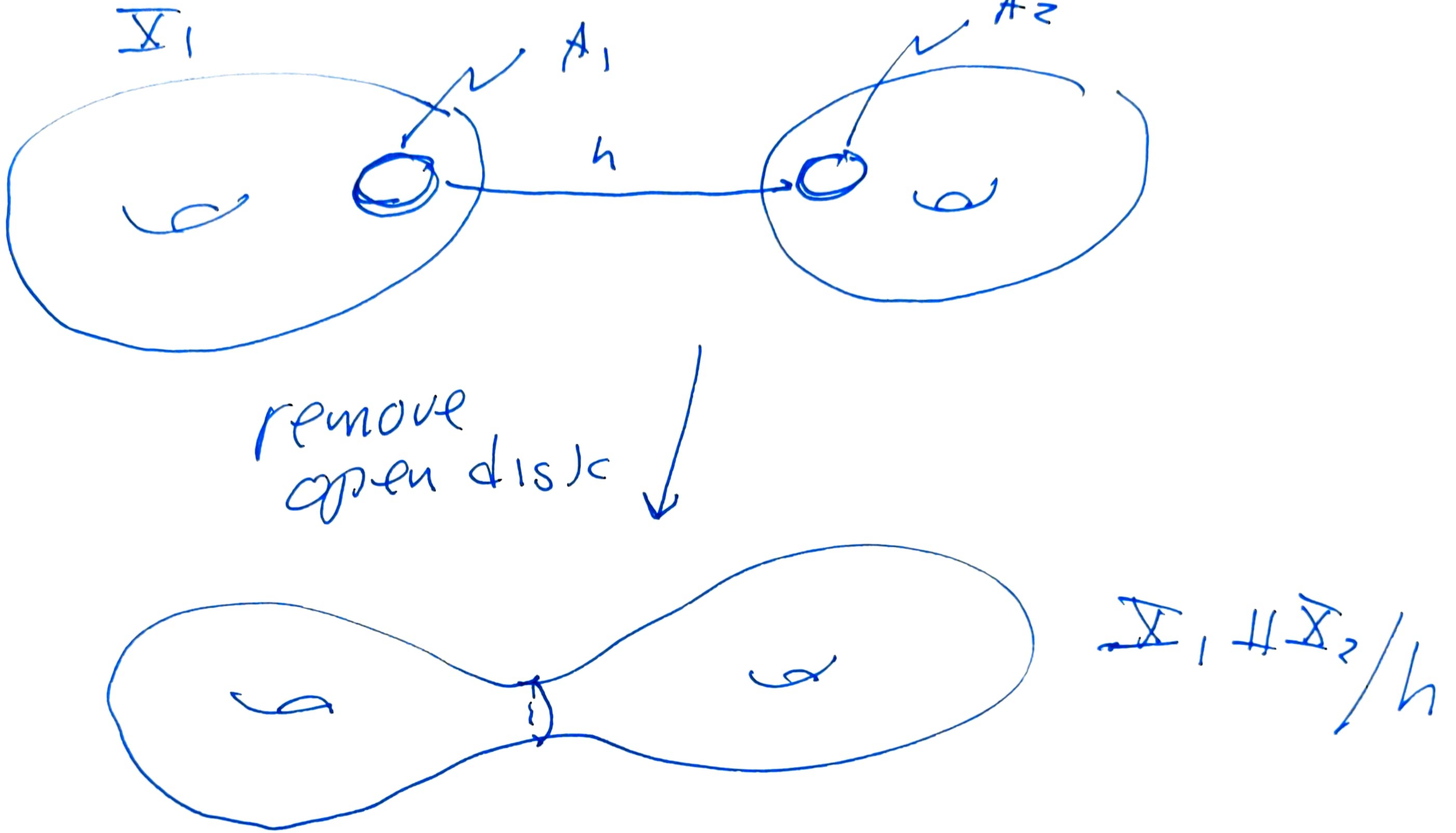
$\Rightarrow$ $X_1 \sqcup X_2 / h$ is the wedge of the two spaces

e.g. S^1 $a_0 = 1$ $\quad S^1$ $b_0 = 1$



$S^2 \vee S^2$





connected sum