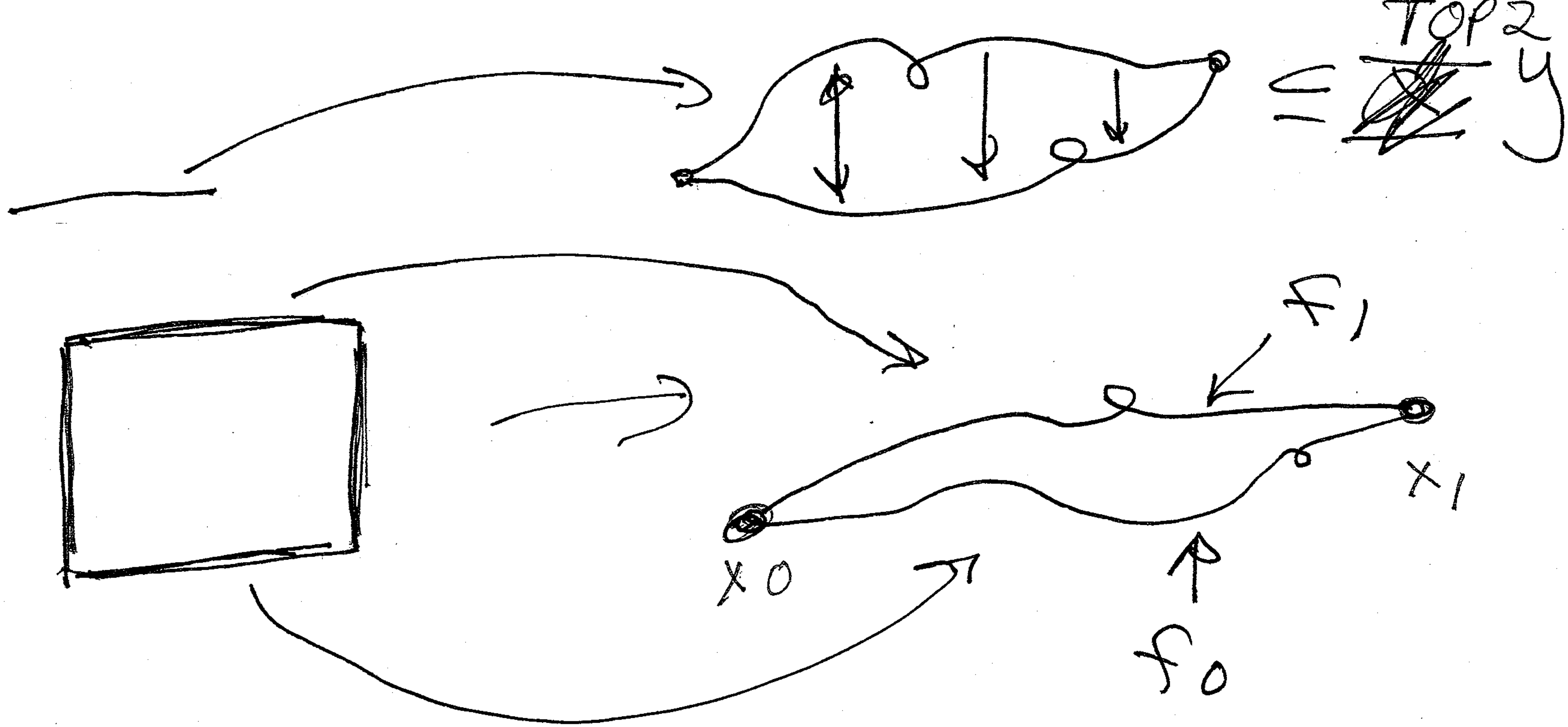


$f_0, f_1 : \underline{I} \to \underline{X} \quad \exists\, f$

$F : \underline{X} \times \underline{I} \to y$

(1) $f_t : f_0 \simeq f_1$

$F(x,0) = f_0$

$F(x,1) = f_1$

$f_0 \underset{p}{\simeq} f_1$

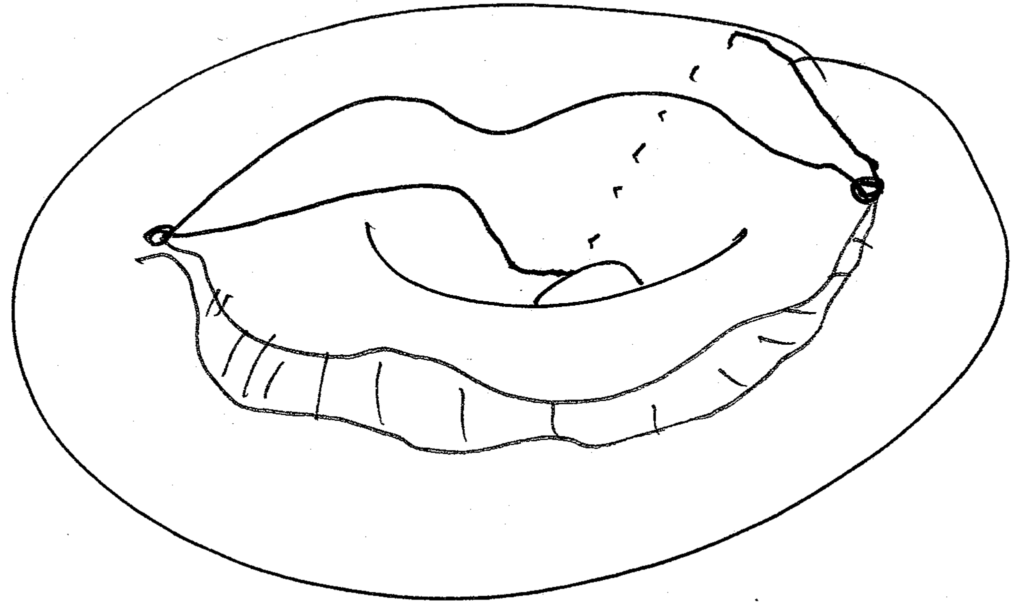
(2) $f_0(0) = x_0 = f_1(0)$

$f_0(1) = x_1 = f_1(1)$

$F(0,t) = x_0$

$F(1,t) = x_1$

$\forall\, t \in \underline{I}$



$f_0 \not\simeq_p f_1$

$f_0 \simeq_p f_1$ is also read f_0 is homotopic to f_1 rel endpoints

(rel = relative)

$P(x_0, x_1) =$ all paths from x_0 to x_1 in X

<u>**Lemma:**</u> $\simeq_p$ is an equivalence relation on $P(x_0, x_1)$

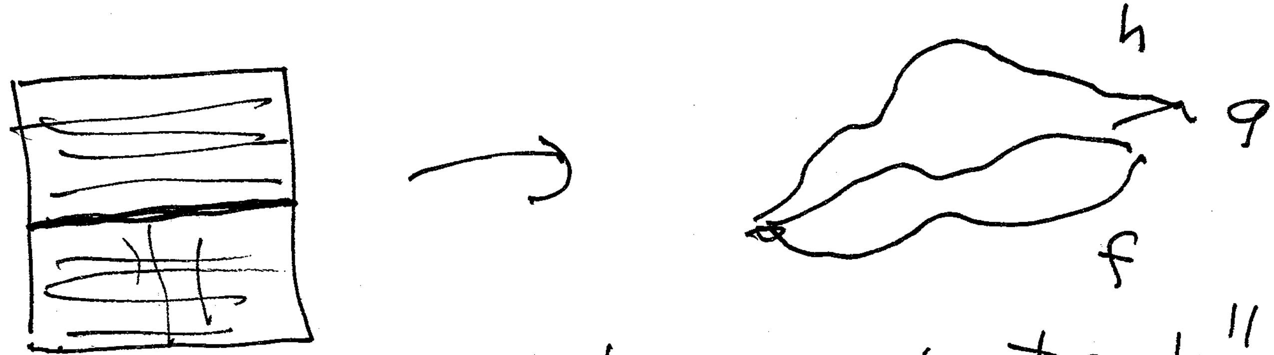
<u>**Proof:**</u> (1) $f \underset{p}{\simeq} f$ via <u>id</u>

(2) $f_t : f_0 \simeq f_1 \Rightarrow f_{(1-t)} : f_1 \simeq f_0$

(3) transitive.

$F: f \simeq_p g \qquad G: g \simeq_p h$

$$H(x,t) = \begin{cases} F(x, 2t) & t \in [0, 1/2] \\ G(x, 2t-1) & t \in [\tfrac{1}{2}, 1] \end{cases}$$

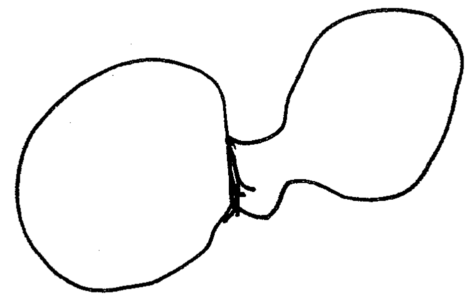


"shrink and stack"

Continuous by the "Pasting Lemma."

Pasting Lemma: $X = A \cup B$

A, B closed



$f : A \to Y$

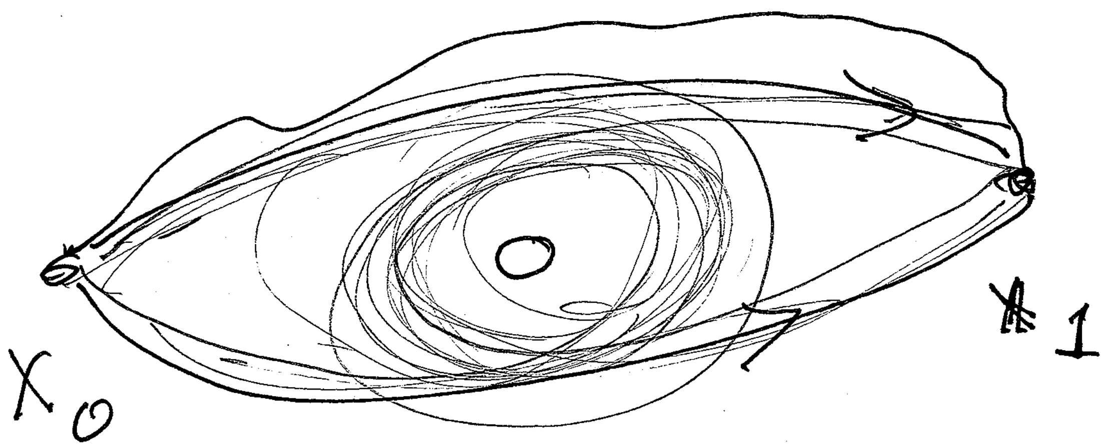
$g : B \to Y$

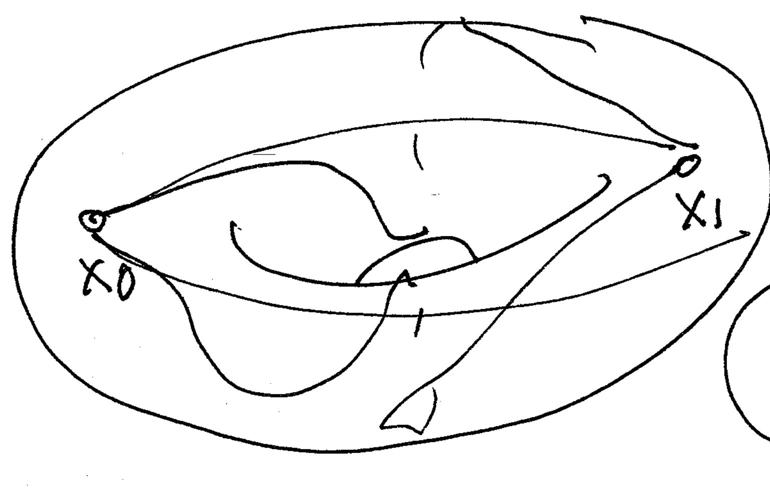
$f|_{A \cap B} = g|_{A \cap B}$

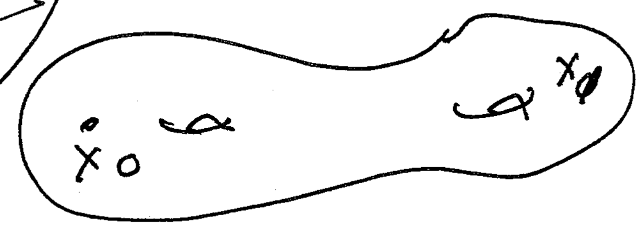
$\Rightarrow \exists\, h : X \to Y$

with $h|_A = f \quad h|_B = g.$

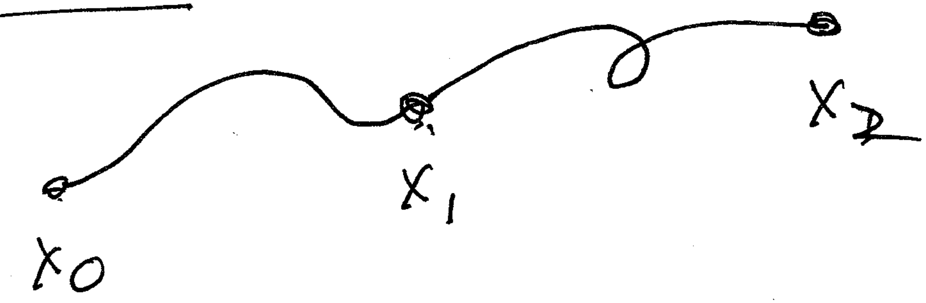
How many equivalence classes in $P(x_0, x_1)$ in $\mathbb{R}^2 - \{\vec{0}\}$







Binary operation on paths



$$*: P(x_0, x_1) \times P(x_1, x_2) \to P(x_0, x_2)$$

$$f * g \in P(x_0, x_2)$$

$$f * g(s) = \begin{cases} f(2s) & s \in [0, 1/2] \\ g(2s-1) & s \in [1/2, 1] \end{cases}$$

cont by pasting since $f(1) = g(0)$

NOTATION: $\overline{P}(x_0, x_1)$ = equiv classes of $\simeq_p$ is $P(x_0, x_1)$

and $[f]$ = equiv class of f.

= path homotopy class.

DEF: $[f] * [g] = [f * g]$

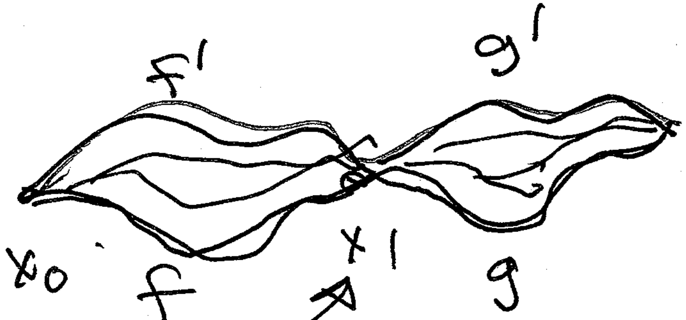
$* : \overline{P}(x_0, x_1) \times \overline{P}(x_1, x_2) \to \overline{P}(x_0, x_2)$.

FACT: This is well-defined

or $f \underset{p}{\simeq} f'$ $\quad g \underset{p}{\simeq} g'$

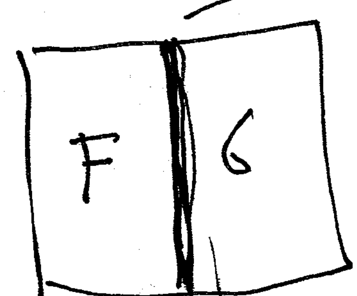
$\Longrightarrow (f * g) \underset{p}{\simeq} (f' * g')$

Proof



$F: f \simeq f'$

$G: g \simeq g'$



$$H(s,t) = \begin{cases} F(2s,t) & s \in [0, 1/2] \\ G(2s-1,t) & t \in [1/2, 1] \end{cases}$$

continuous ~~since~~ ~~F(~~ since agree on overlap.

Theorem $\ast : \overline{P}(x_0, x_1) \times \overline{P}(x_1, x_2) \to \overline{P}(x_0, x_2)$ has the following properties

(1) Associative.

$$[f] \ast ([g] \ast [h]) = ([f] \ast [g]) \ast [h]$$

(2) Let $e_x : I \to \{x\}$ constant path

$$[f] \ast [e_{x_1}] = [f]$$

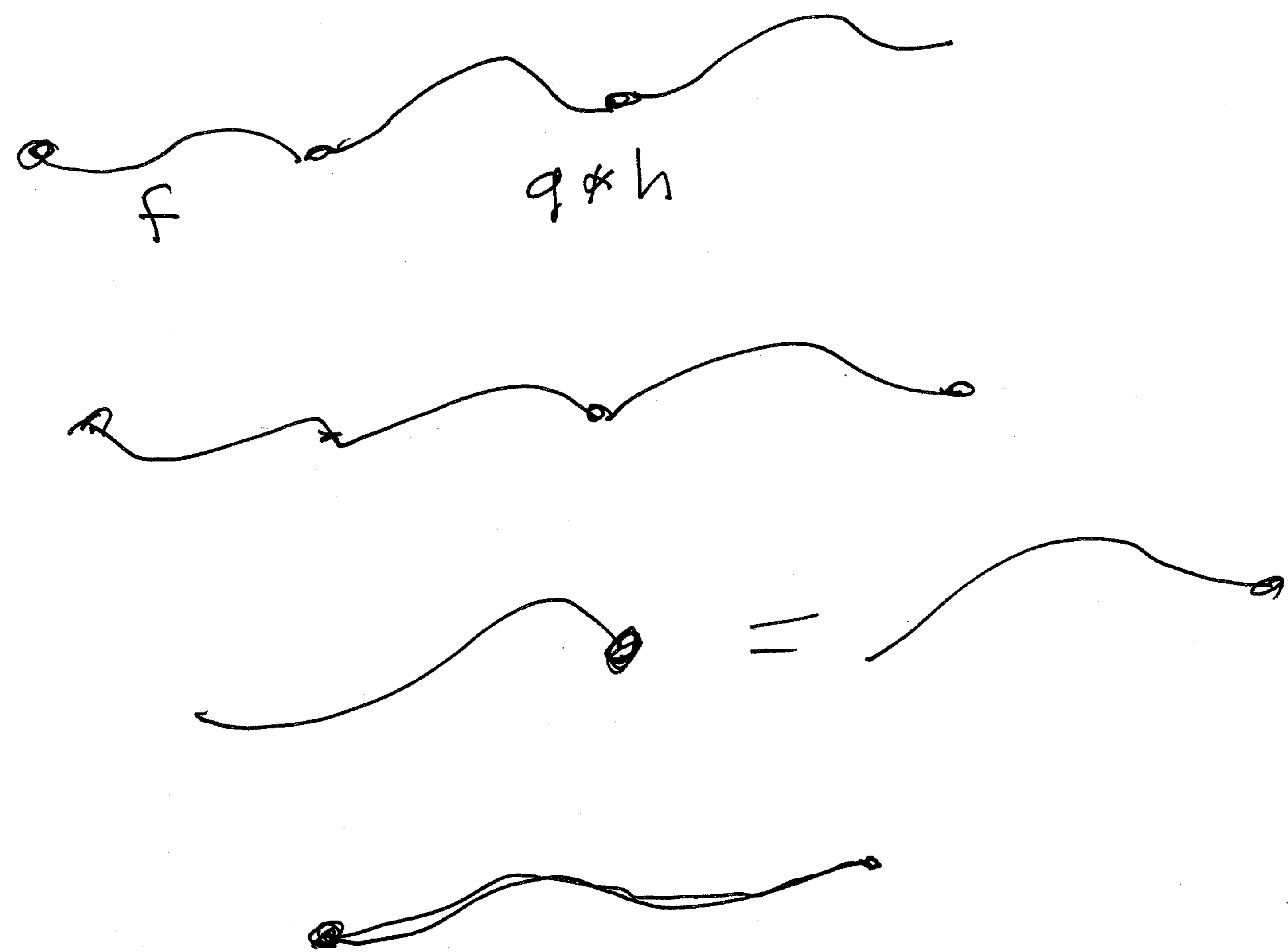
$$[e_{x_0}] \ast [f] = [f]$$

(3) Let $f^{-1}(x) = f(1-x)$

$$\Rightarrow \quad [f] \ast [f^{-1}] = [e_{x_0}]$$

$$[f^{-1}] \ast [f] = [e_{x_1}]$$

Pictures



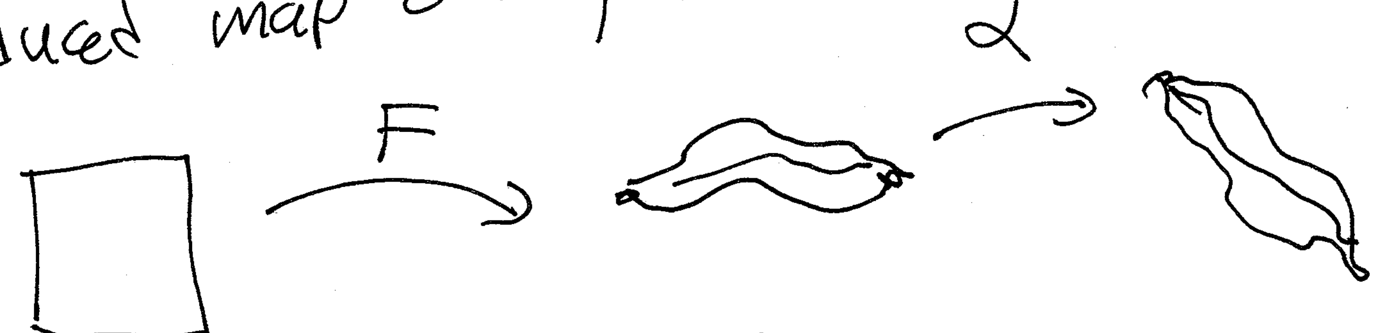
Prelminaries

$f, f' : I \to X$

$\alpha : X \to Y$ continuous

If $f \underset{p}{\simeq} f' \Rightarrow \alpha \circ f \underset{p}{\simeq} \alpha \circ f'$

"induced map on paths"



$\alpha \circ F : \alpha \circ f \simeq \alpha \circ f'$

$[f] \in \overline{P}(x_0, x_1) \quad x_0, x_1 \in X$

$\alpha : X \to Y$ cont

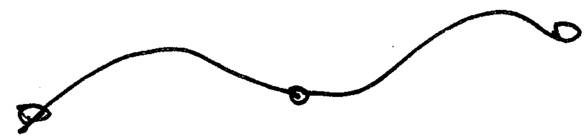
$\alpha_* [f] \in \overline{P}(\alpha(x_0), \alpha(x_1))$

via $\alpha_* [f] = [\alpha \circ f]$

is well defined

$\alpha_* : \overline{P}(x_0, x_1) \to \overline{P}(\alpha(x_0), \alpha(x_1))$

Prelim 2



$f, g : I \to X$

$f(1) = g(0)$

$\alpha : X \to Y$

$\alpha \circ (f * g) = (\alpha \circ f) * (\alpha \circ g)$

$\boxed{\text{homomorphism}}$

how the product on X get pushed to the product on Y.

Proof of (2):

$$\left.\begin{array}{l} \text{Define } e_0 : I \to \{0\} \\ i : I \to I \end{array}\right\} \text{paths in } I = [0,1]$$

so $e_0 * i$ is also path in I

<u>and</u> I is convex so

$$e_0 * i \underset{p}{\simeq} i \qquad f : I \to X$$

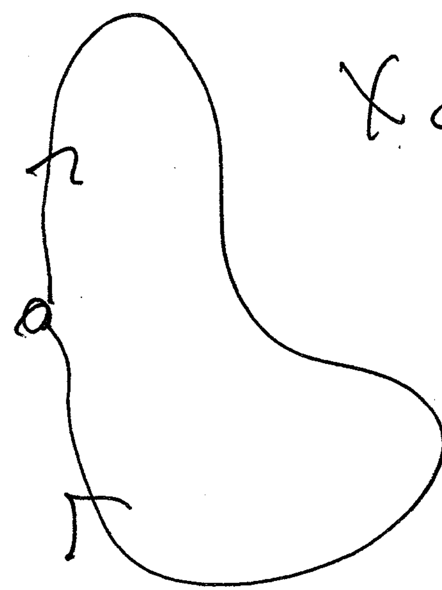
Then prelim 1 $\quad f \circ (e_0 * i) \underset{p}{\simeq} f \circ i$

prelim 2 $\quad (f \circ e_0) * (f \circ i) \underset{p}{\simeq} f \circ i$

$$e_{x_0} * f \underset{p}{\simeq} f$$

so $[e_{x_0}] * [f] = [f]$

Proof of (3) next time



$x_0 = x_1$