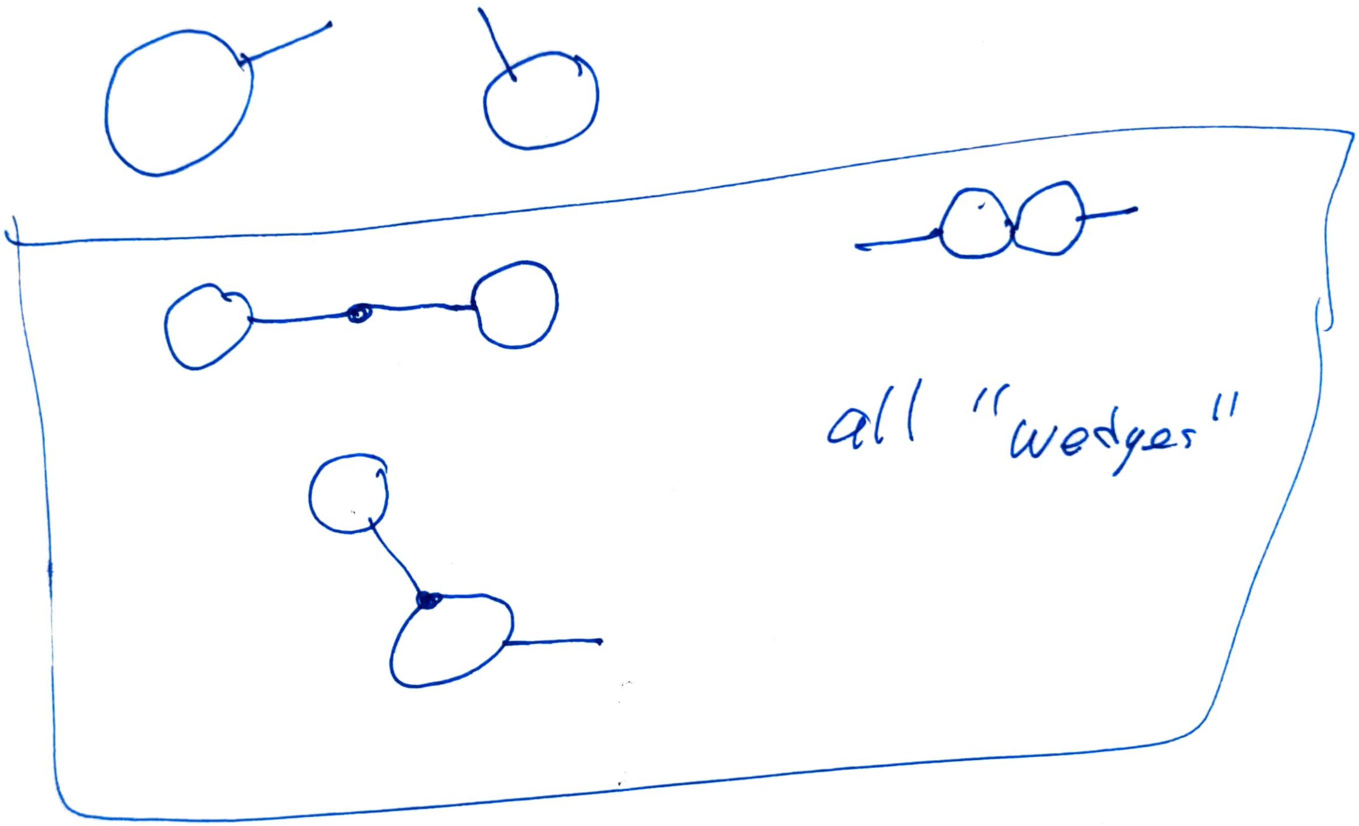
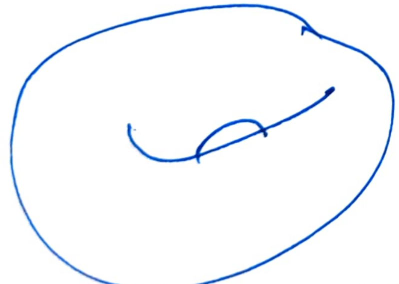
Comments on last lecture.

1. Wedge can depend on points chosen



It doesn't depend on the point when the space are what's called homogeneous

i.e. $\forall x, y \in X$ $\exists$ $h: X \to X$ a homeomorphism with $h(x) = y$. - so "space looks the same everywhere"



homogeneous

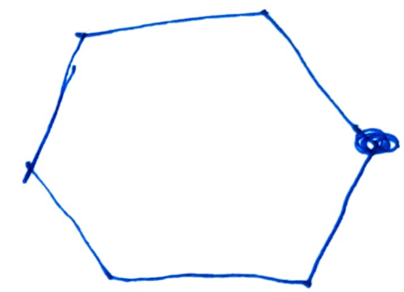
not homogeneous

In topolg $X \vee Y$ X wedge Y

In Diff Geom $\eta \wedge \eta'$ – wedge product of differential forms

Polygonal identifications and Surfaces

P_n = regular n-gon in $\mathbb{R}^2$ and assume $(1,0)$ is a vertex a labeling of P_n is a word.

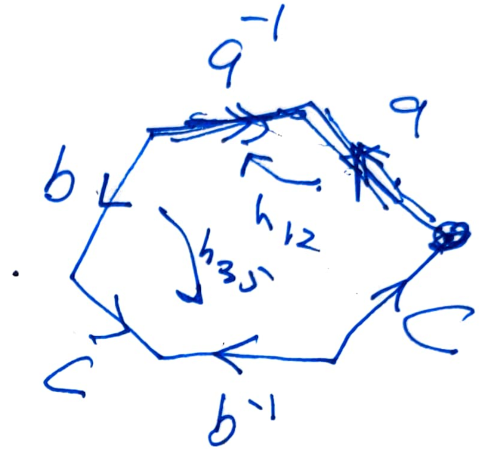


$$W = a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \cdots a_{i_n}^{\varepsilon_n}$$

~~with~~ with each a_i an arb symbol and $\varepsilon_i = \pm 1$

We label the edges of P_n starting in order from $(1,0)$ as $a_{i_j}^{\varepsilon_j}$ when $\varepsilon_j = 1$ is counterclockwise and $\varepsilon_j = -1$ is clockwise.

eg// $W = aa^{-1}bcb^{-1}c$



For each pair of edges e_i and e_j with the same symbol. Define $h_{ij}: e_i \to e_j$ a homeomorphism which respects orientation

Define an equiv relation on P_n

(1) $x \sim x \quad \forall x \in P_n$

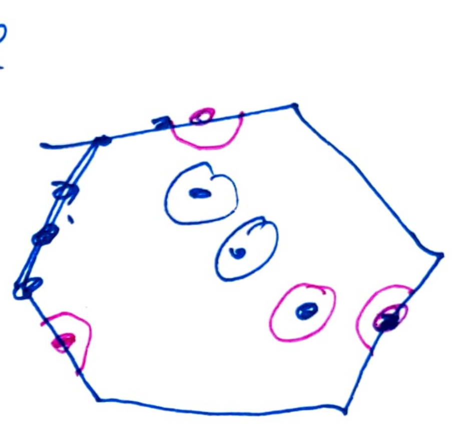
(2) If e_i and e_j have the same symbol ~~identif~~ $x \sim h_{ij}(x)$

Let $X = P_n/_h = P_n/W$ be the quotient space with the quotient topology.

RK: cyclic permutation of W yields a homeomorphic quotient

__Lemma__ P_n/W is Hausdorff.

__Proof__: case by case



(1) Both interior

(2) one on interior or
edge on interior

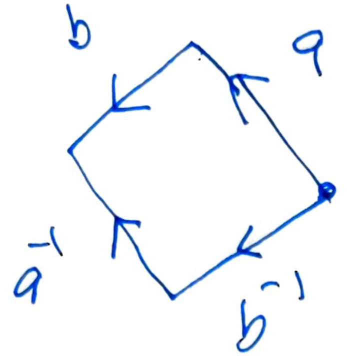
__etc__

$\boxed{\text{The fundamental group of } P_n/W}$

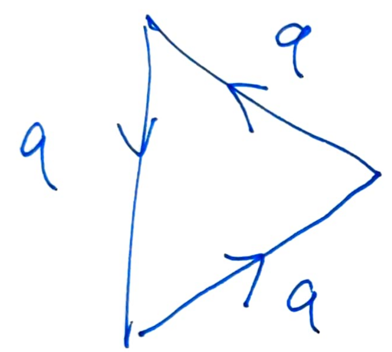
Theorem Assume that $W = a_{i_1}^{\varepsilon_1} \cdots a_{i_n}^{\varepsilon_n}$ is such that all the vertices of P_n are identified with a single point in P_n/W.

If $\{a_1, \ldots, a_k\}$ are the distinct lables that occur in $W \Rightarrow$

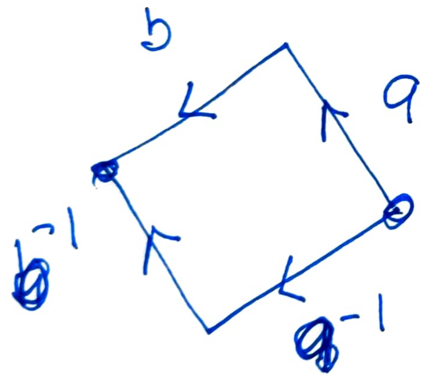
$$\pi_1(X) \cong \langle \alpha_1, \ldots, \alpha_k \mid \alpha_{i_1}^{\varepsilon_1} \alpha_{i_2}^{\varepsilon_2} \cdots \alpha_{i_n}^{\varepsilon_n} \rangle$$



$\pi_1(X) = \langle a, b : a b a^{-1} b^{-1} \rangle$



$\pi_1(X) = \langle a : a^3 \rangle = \mathbb{Z}_3$



~~$\pi_1(X) = \langle a, b : a b b^{-1} a^{-1} \rangle$~~

~~$= \langle a, b \mid e \rangle$~~

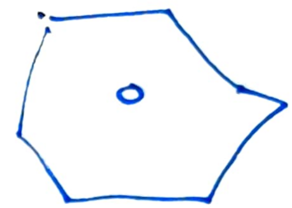
S^2 $\pi_1(S^2) = e$

vertices go to disjoint points.

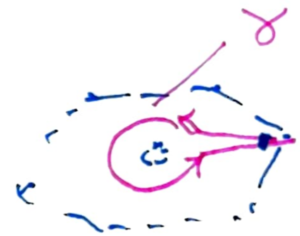
Proof via SVK

Let $U = P_n - \{0\}/W$

$V = (P_n - Bd(P_n))/W$



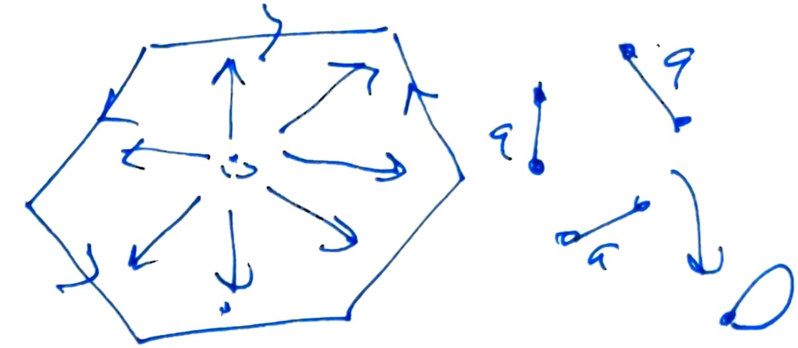
$U \cap V = P_n - (Bd(P_n) \cup \{0\})$



Let $x_0 = (1, \tilde{0})/W$

$\pi_1(U \cap V) \cong \mathbb{Z}$ since it has a circle as as deformation retract with generator γ as pictured

U def retracts onto $\mathrm{Bd}(P_n)/W$



$\mathrm{Bd}(P_n)/W$ is the wedge of k-circles since there are k disjoint label

so $\pi_1(U, x_0) = \langle \alpha_1, \ldots, \alpha_k \rangle$

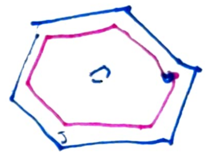
each α_i ~~for~~ from the projection of the edges with symbol a_i

$\pi_1(V, x_0) = \langle e \rangle$

$i_1 : \pi_1(U \cap V) \to \pi_1(U)$
$i_2 : \pi_1(U \cap V) \to \pi_1(V)$

$(L_2)\ (\gamma) = e$, $\forall$ iss.c.

$L_1(\gamma) = \alpha_{L_1}^{\varepsilon_1} \cdots \alpha_{L_n}^{\varepsilon_n}$



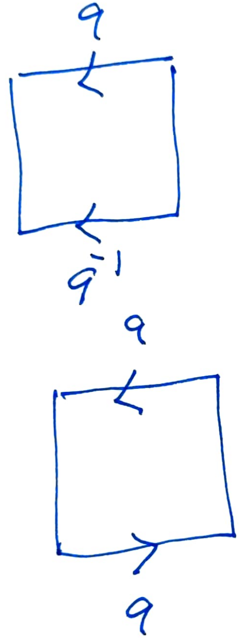
By SVK

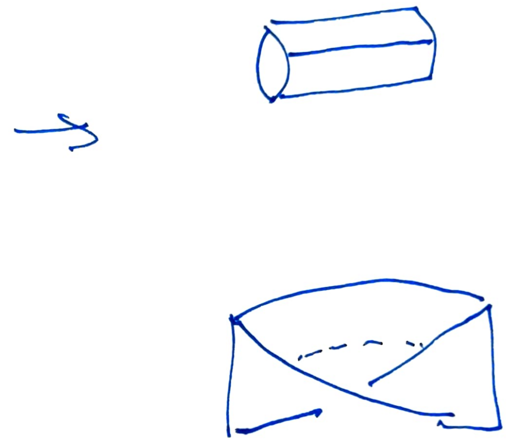
$\Rightarrow \pi_1(X, x_0) =$

$\langle \alpha_1, \dots, \alpha_k : \alpha_{L_1}^{\varepsilon_1} \cdots \alpha_{L_n}^{\varepsilon_1} \rangle$

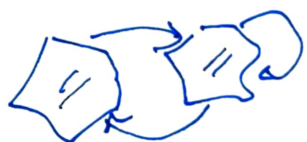
<u>Remarks</u>: There are many variants of the main construction.

① You can allow unlabeled edges which become boundary.

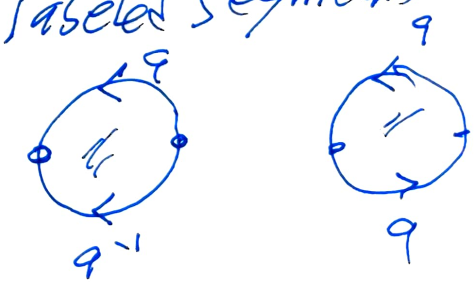




② multiple polygons



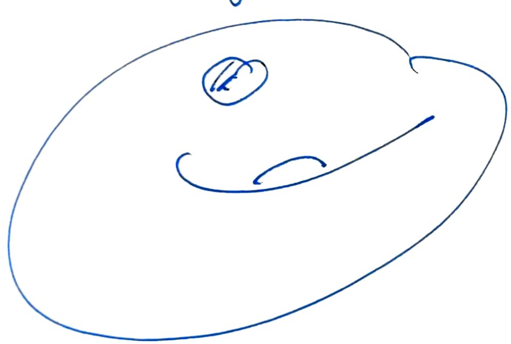
③ Sometimes you use labeled segments on the boundary of D^2



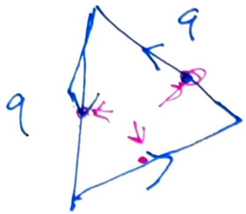
Def: X is a n-dimensional topological manifold if it is Hausdorff and each $x \in X$ has a neighborhood homeomorphic to

$\cancel{\mathring{B}}$ $\mathring{B}^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$

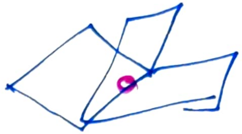
open unit ball in n-dimensions.



DEF. Surface $=$ 2-dim top mfd



is P_3/W a surface?

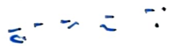


not a surface

(yes, branched surface)

any open set in $\mathbb{R}^2$ is a surface

$\mathbb{R}^2$ - Cantor set



We are interested in compact surfaces