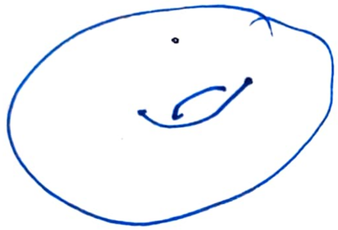
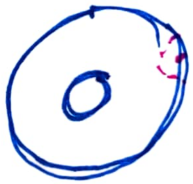
Def: A surface Σ is a HD space and 2.2/

$\forall x \in \Sigma$, $\exists$ an open nhd homeomorphic to $\mathring{B}^2 = \{x \in \mathbb{R}^2 : \|x\| < 1\}$



$\mathring{B}^2$, open sets in the plane

not



closed annulus
surface with boundary.

Connected sum of surfaces

$h \wedge \qquad i^{-1} \wedge . c^{-1}$

Given two connected surfaces Σ_1 and Σ_2
pick points $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$
with open disk neighborhoods B_1 B_2
with boundary circles ~~B~~ S_1 and S_2

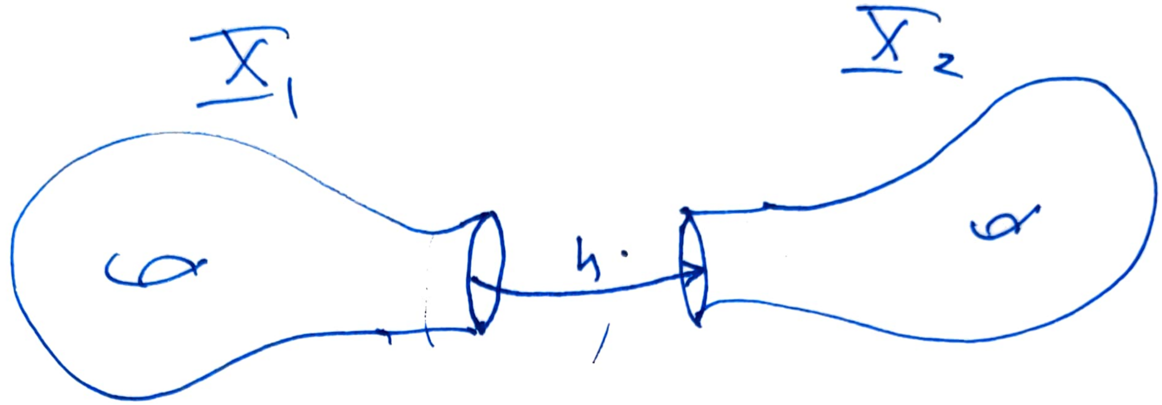
$$X_1 = \Sigma_1 - \mathring{B}_1$$

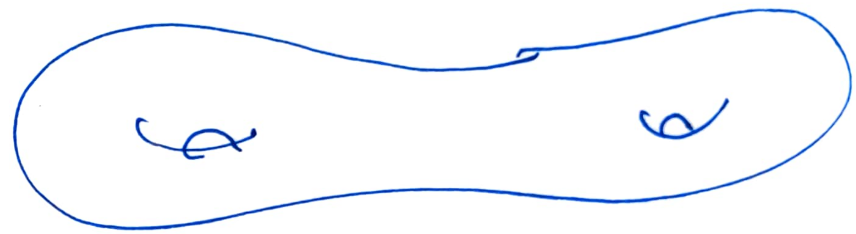
$$X_2 = \Sigma_2 - \mathring{B}_2$$

The connected sum

$$X_1 \# X_2 = (X_1 \amalg X_2)/h$$

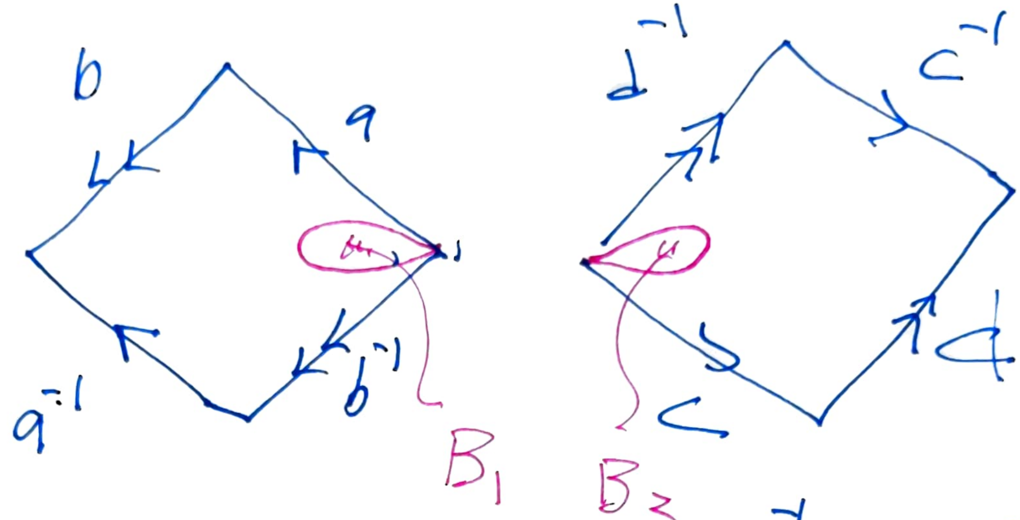
where $h: S_1 \to S_2$ is a homeomorphism.



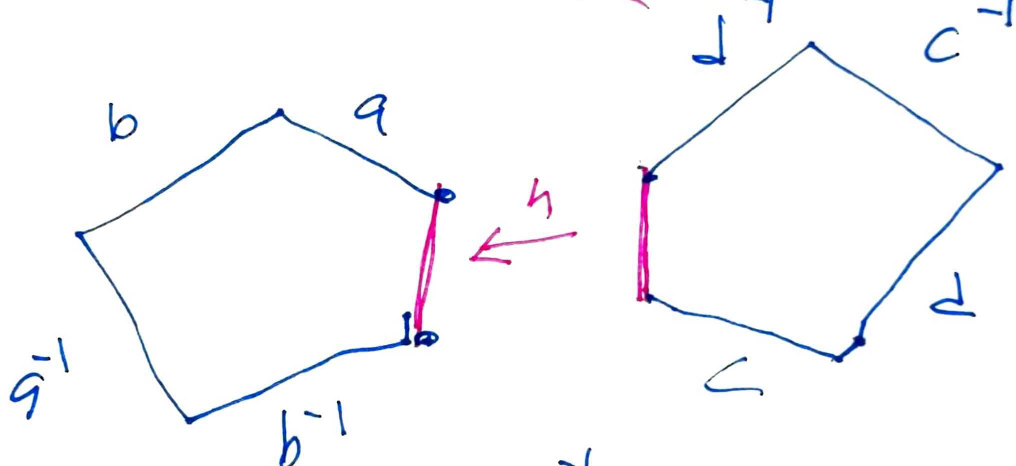
$\Sigma_1 \# \Sigma_2 =$ 

What does this do to π_1?

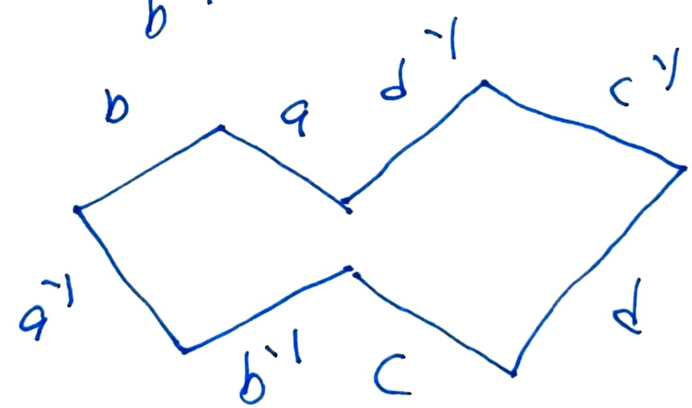
We use labeled polygons.



$T^2 \sqcup T^2$



$X_1 \sqcup X_2$

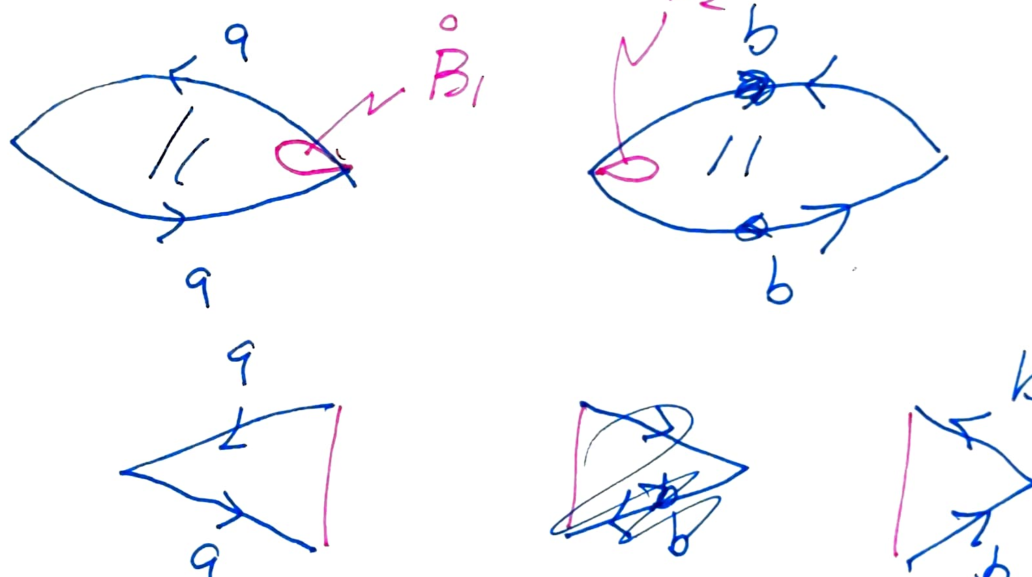


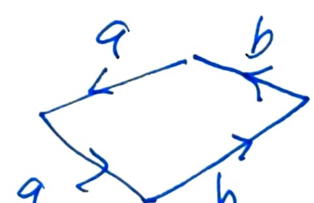
$T^2 \# T^2$

So $\pi_1(\mathbb{T}^2 \# \mathbb{T}^2) = \langle a, b, c, d : aba^{-1}b^{-1}\overline{cdc^{-1}d^{-1}} \rangle$

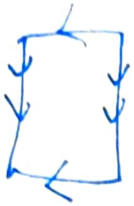
<u>NOTATION</u> $[a,b] = aba^{-1}b^{-1}$ commutator

PP # PP





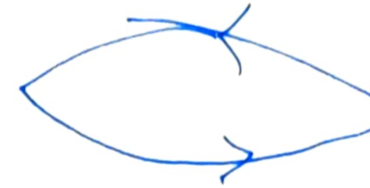
$\pi_1(PP \# PP)$
$= \langle a, b : a^2 b^2 \rangle$



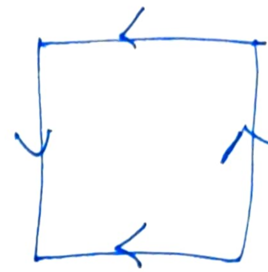
$\mathbb{T}^2$

$\boxed{\text{orientable}}$

$\mathbb{P}\mathbb{P}$



K = Klein bottle.



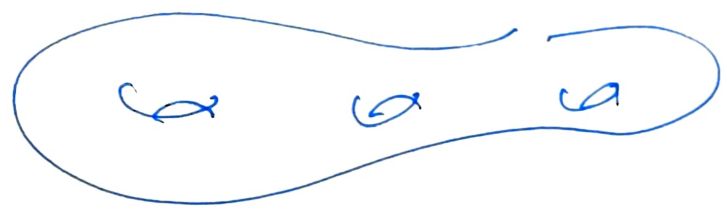
don't embed in $\mathbb{R}^3$

$\boxed{\text{non-orientable}}$

Classification of compact surfaces

(1) If M is orientable $\Longrightarrow$

M is homeomorphic to the connected sum of n-tori for some $n > 0$ or a sphere. And n is called the genus of M



$n=3$

and $\pi_1(M) = \langle a_1, b_1, \ldots, a_n, b_n : [a_1,b_1] \cdots [a_n,b_n] \rangle$

2 If M is non-orientable $\Rightarrow$

M is homeomorphic to ~~(n)~~ the connected sum of n-projective planes for sum $n \geq 1$. In this case,

$$\pi_1(M) = \langle a_1, \ldots, a_n : a_1^2 a_2^2 \ldots a_n^2 \rangle$$

Further, If n is odd then M is homeomorphic to the direct sum of $\frac{1}{2}(n-1)$ tori and a projective plane.

If n is even then M is homeomorphic to the connected sum of $\frac{1}{2}(n-2)$ tori and a Klein bottle.

Finally, all these cases are non-homeomorphic to each other.

eg $T^2 \# \cdots \# T^3$ n times is not homeo to $T^2 \# \cdots \# T^2$ m-times $n \neq m$.

This last result is proved via Abelianization

DEF: F is a group, $[F,F]$ (F') is the commutator subgroup generated by the commutators. (It is normal) and

$$F/[F,F]$$ is the Abelianization

of F.

The abelianization of $\pi_1(X, x_0)$ is called $H_1(X;\mathbb{Z})$ the first homology group (often defined differently)

$$H_1(X;\mathbb{Z}) = \pi_1(X,x_0) \big/ [\pi_1(X,x_0), \pi_1(X,x_0)]$$

<u>Lemma</u> $H_1(n\text{-connected sum of } T^2) \cong \mathbb{Z}^{2n}$

$H_1(m\text{-connect sum of } PP) \cong \mathbb{Z}^{m-1} \oplus \mathbb{Z}_2$

all non-isomorphic

eg

$$\langle a \mid a^2 \rangle = \mathbb{Z}_2$$

$$\boxed{n = 1}$$