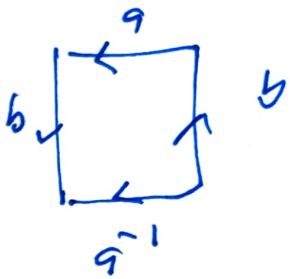
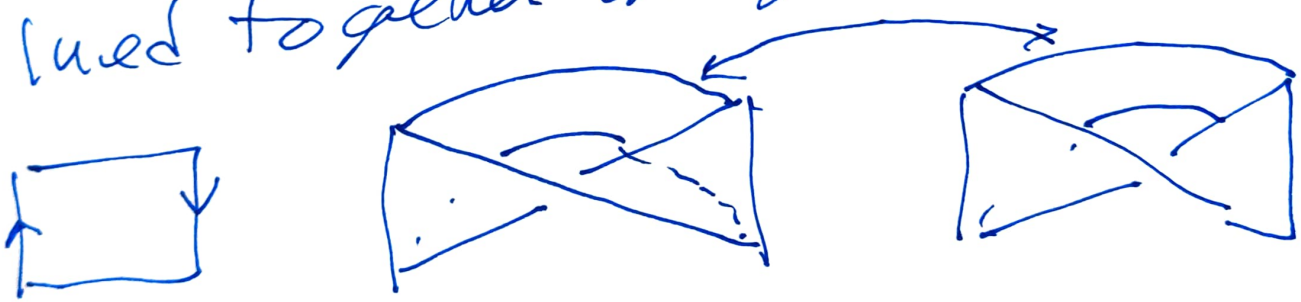
Proposition: $PP \# PP \cong K$

so $\langle a, b : a^2 b^2 \rangle \cong \langle a, b : a b a^{-1} b \rangle$

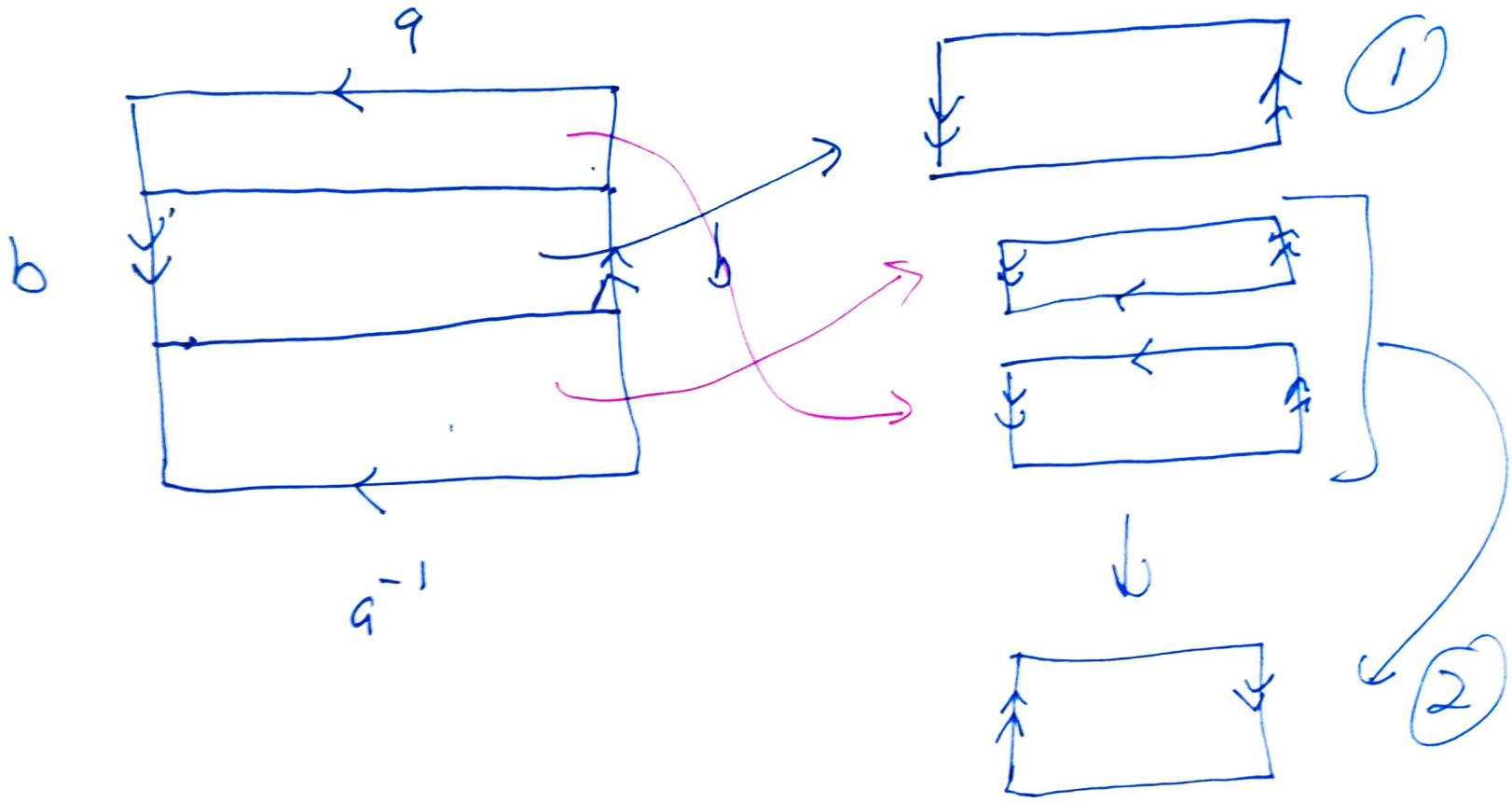


Proof: "Cutting and pasting"

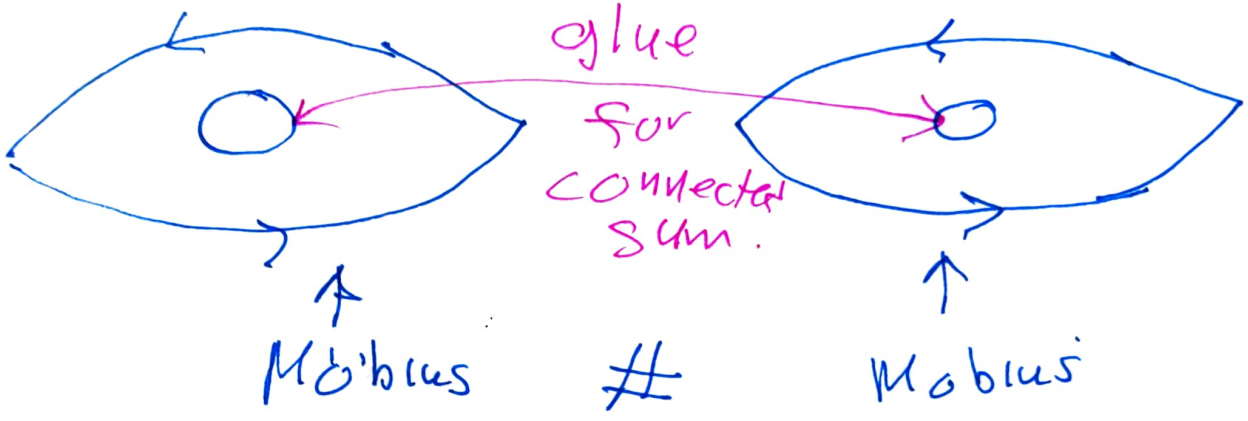
Strategy: Both $PP \# PP$ and K are homeomorphic to two Möbius bands glued together along their boundary



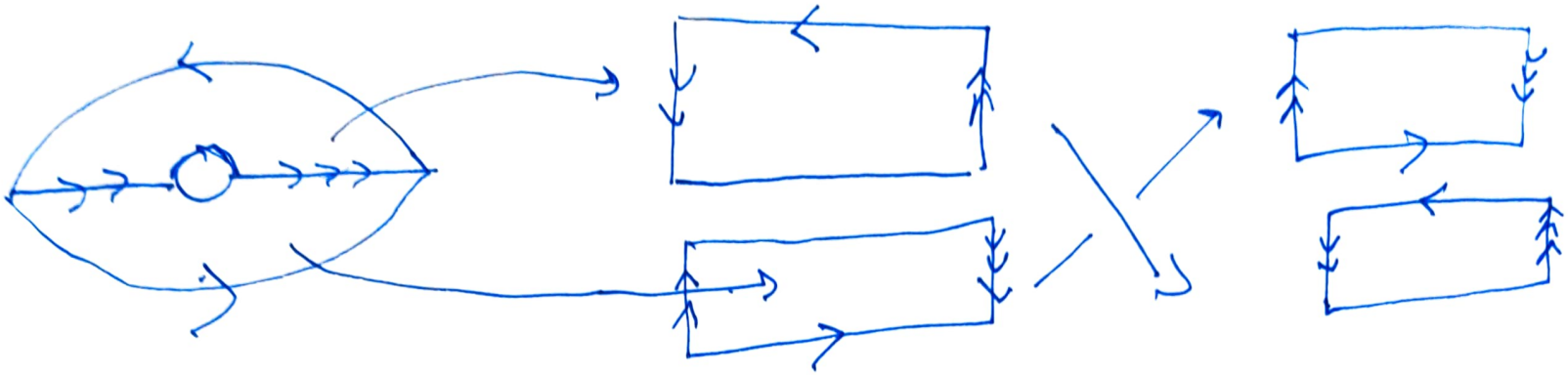
First K: Show we can cut it along a circle and get two Möbius



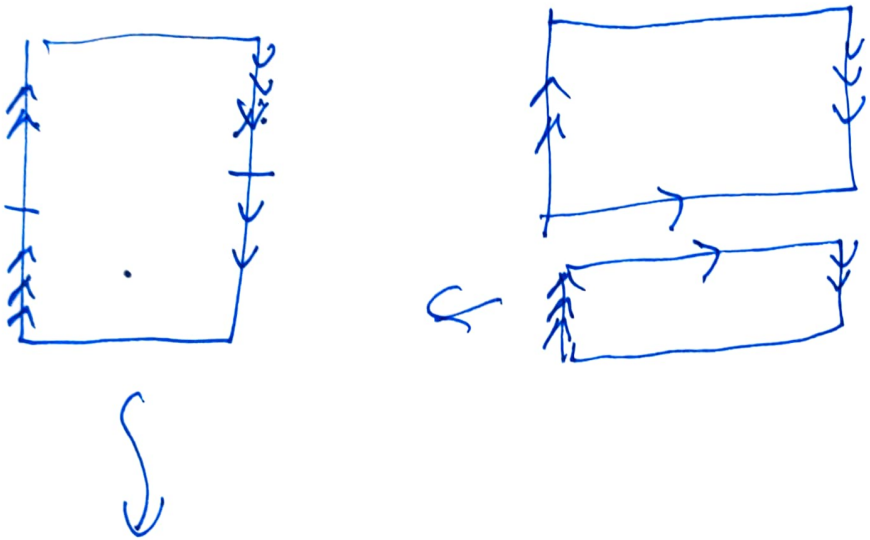
(2) Show $\mathbb{P}\mathbb{P} - \{\text{open disk}\}$ is a Möbius band



Then $\mathbb{P}\mathbb{P} \# \mathbb{P}\mathbb{P}$ is two glued Möbius bands.



flip lower



Möbius band.

Recall: M is compact surface

(1) M is orientable:

$$M \cong \mathbb{T}^2 \# \dots \# \mathbb{T}^2 \quad n\text{-times}$$

$$\pi_1(M) \cong \langle a_1, b_1, \dots a_n, b_n : [a_1, b_1] \dots [a_n, b_n] \rangle$$

$$H_1(M) \cong \mathbb{Z}^{2n}$$

Call this M, M_n^+

(2) M is non-orientable.

$$M \cong \mathbb{P}\mathbb{P} \# \dots \# \mathbb{P}\mathbb{P} \quad m\text{-times}$$

$$\pi_1(M) = \langle a_1, a_2, \dots, a_m : a_1^2 a_2^2 \dots a_m^2 \rangle$$

$$H_1(M) = \mathbb{Z}^{m-1} \oplus \mathbb{Z}_2$$

Given F, $F/[F,F]$ is F^{ab}, the abelianization

$\boxed{\text{Algebra lemma}}$ $N \triangleleft F$ (N is normal subgroup)

$p: F \to F/[F,F]$ then

$$\frac{F/N}{[F/N : F/N]} \cong \frac{F/[F,F]}{p(N)}$$

"The abelianization of the quotient is the quotient of the abelianizations".

CORR: F is free on $\alpha_1, \ldots, \alpha_n$.

$x \in F$ (the relation) and N is the least normal subgroup containing x.

$$\Rightarrow \quad \frac{F/N}{[F/N, F/N]} \cong \frac{F/[F,F]}{W(x)}$$

where W is the subgroup generated by $p(x)$.

Proof Since N is generated by x and its conjugates $p(N) \leq$ Abelian group is generated by $p(x)$: Note $F/[F,F]$ = Free abelian on $p(\alpha_1), \ldots, p(\alpha_n)$

Homework

(1) $\pi_1(M_n^+) = \langle \cancel{a_1} \rangle$. Free group on $a_1, b_1, a_n, b_n \big/ N$

with N smallest normal group contain $[a_1, b_1] \cdots [a_n, b_n]$

By ~~coh.~~ cor.

$$H_1(M_n^+) = F_{2n}/N \cong \frac{\mathbb{Z}^{2n}}{\text{group gen by } [\alpha_1, \beta_1] \cdots [\alpha_n, \beta_n] \text{ in } \mathbb{Z}^{2n}} = \mathbb{Z}^{2n}$$

but $[\alpha_1, \beta_1] = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1}$

in $\mathbb{Z}^n$ $p(\ ) = \alpha_1 + \beta_1 - \alpha_1 - \beta_1 = 0$

(2) $\pi_1(M_n^-) = \langle a_1, \dots, a_n : a_1^2 a_2^2 \dots a_n^2 \rangle$

same logic $H_1 \cong \mathbb{Z}^n / V$

where V is the group generated by
$p(a_1^2 a_2^2 \dots a_n^2) = 2(\alpha_1 + \dots + \alpha_n)$

Treat H_1 as a $\mathbb{Z}$-module and as an linear algebra - change the basis to
$\alpha_1, \dots, \alpha_{n-1}, \beta$ where $\beta = \alpha_1 + \dots + \alpha_n$

(basis = set of generators)

Then

$$H_1 \cong \frac{\text{Free abelian on } (\alpha_1, \ldots, \alpha_{n-1}, \beta)}{\text{Subgroup generated by } 2\beta}$$

$$= \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$= \mathbb{Z}^{n-1} \oplus \mathbb{Z}_2 .$$

$\Rightarrow$ all $H_1(M_n^+)$ and $H_1(M_m^-)$ are non-isomorphic and so all $\pi_1(M_n^+)$ and $\pi_1(M_m^-)$ are non-isomorphic and so all M_n^+ and M_m^- are not homotopy equiv.