Tychonoff Th'm:

"Arbitrary product of compact spaces is compact in the product topology"

$[0,1]^{\mathbb{Z}}$ is cpt.

$[0,1]^{\mathbb{R}}$ is cpt.

DEF: open covers have finite sub covers
= compact

FACTS:
(1) closed in cpt is cpt
(2) cpt in HD is closed
(3) continuous image of cpt is cpt
(4) finite product of cpt is cpt.

The Finite Intersection Property: (fip).

DEF: $\mathcal{C}$ is a collection of subsets of X ($\mathcal{C} \subseteq P(X)$) It has the fip if for every finite subcollection $C_1, \ldots, C_n$ we have

$$\bigcap_{i=1}^{n} C_i \neq \emptyset.$$

Thm: X is cpt $\Leftrightarrow$ every subcollection of **closed** sets $\{C_\lambda\}_{\lambda \in \Lambda}$ with the fip has $\bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset$.

Eg// $C_n = [n, \infty) \subseteq \mathbb{R}$, closed

$\{C_n\}_{n \in \mathbb{Z}}$ has f.i.p but

$\bigcap_{n \in \mathbb{Z}} C_n = \emptyset$, so $\mathbb{R}$ is not cpt.

Recall Arbitrary products.

$$\prod_{\lambda \in \Lambda} \overline{X}_\lambda = \{ \underline{x} = (x_\lambda)_{\lambda \in \Lambda} : x_\lambda \in \overline{X}_\lambda \}$$

$$= \text{all functions } \underline{x} : \Lambda \to \bigcup_{\lambda \in \Lambda} \overline{X}_\lambda$$

$$\text{such that } \underline{x}(\lambda) \in \overline{X}_\lambda$$

Is this always nonempty.?

$\underline{x}$ is a "choice function"

Independent of usual ZF axioms of set theory. → Axiom of Choice.

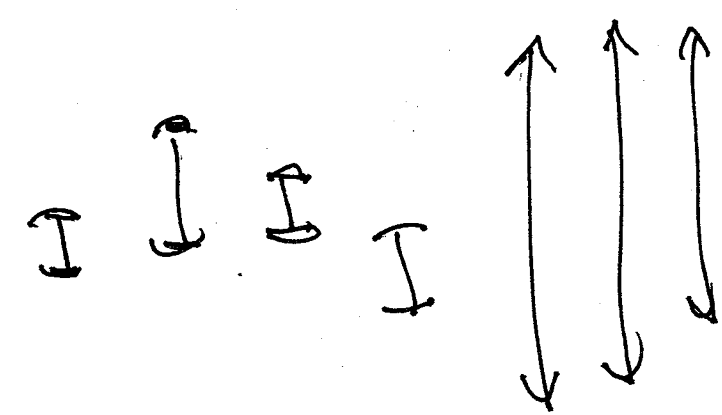
Product Topology has a base

~~projection~~

all $\prod_{\lambda \in \Lambda} U_\lambda$ with U_λ open in X_λ

and $U_\lambda = X_\lambda$ for all but finitely many λ.

Picture in $\mathbb{R}^{\mathbb{N}}$



$\boxed{\text{Ideas in the proof of Tychonoff}}$

<u>Simple idea</u> - use the f.i.p

Try out on just $X_1 \times X_2$

$\mathcal{A} = \{A_\lambda\}$ closed sets in $X_1 \times X_2$ with f.i.p.

$\Rightarrow \{\pi_1(A_\lambda)\}$ is a collection in X_1 with f.i.p. $\Rightarrow$ so is $\overline{\pi_1(A_\lambda)}$ $\Rightarrow$ Since X_1 is cpt so $\exists\ x_1 \in \bigcap \overline{\pi_1(A_\lambda)}$ by

fip for X_1

$\Rightarrow$ Similarly, $\exists\ x_2 \in \bigcap \overline{\pi_2(A_\lambda)}$

$\Rightarrow$ $\boxed{(x_1, x_2) \in \bigcap A_\lambda \text{ in } X_1 \times X_2}$

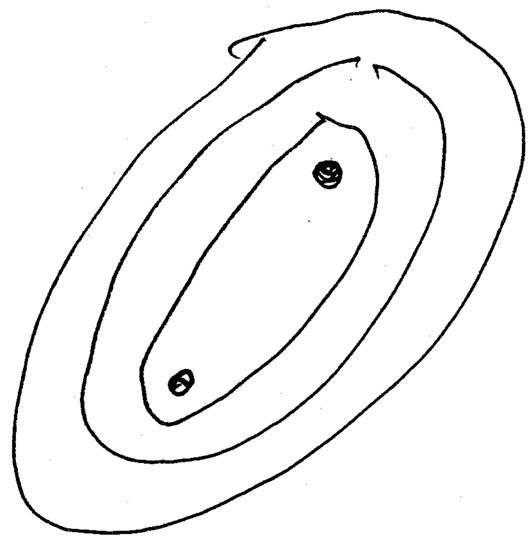
$\Downarrow$

FALSE

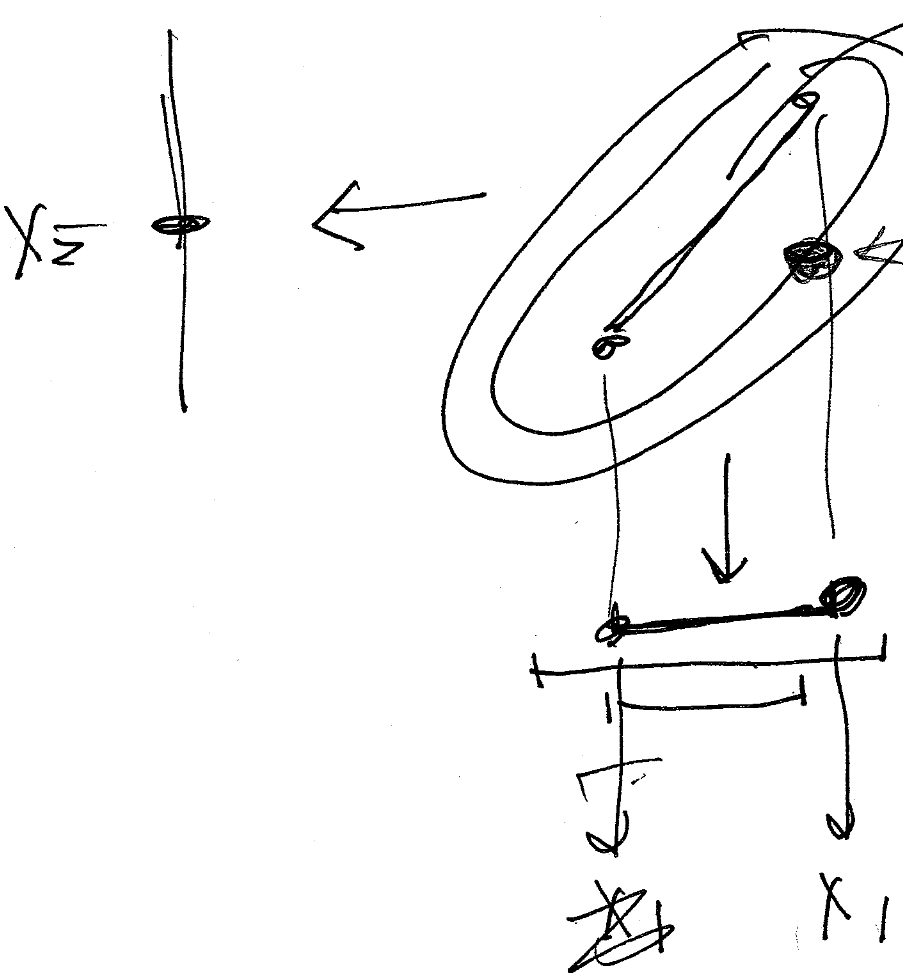
Why is it false?

Example: $X_1 = X_2 = [0,1]$.

$\mathcal{A}$ be all closed elliptical regions with foci $(1/3, 1/3)$ and $(1/2, 2/3)$



$\mathcal{A}$ has fip.



$\bigcap \pi_1(A_x) = [1/3, 1/2]$

pick $x_1 = 1/2$

$\bigcap \pi_2(A_x) = [\frac{1}{3}, \frac{2}{3}]$

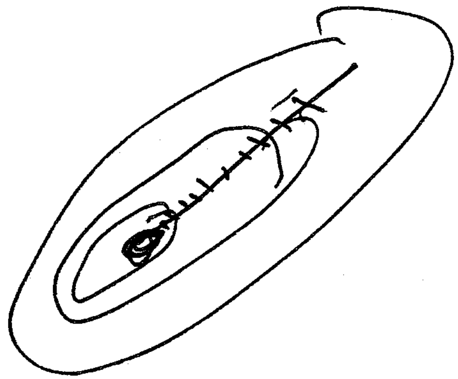
$x_2 = 1/2$

$(\frac{1}{2}, \frac{1}{2}) = (x_1, x_2) \notin \bigcap A_x$

We made a bad choice for x_1, x_2 but it isn't clear, in general, how to make a good choice.

- we ensure a good choice be enlarging the set $\star$.

Let $\mathcal{D}$ be all elliptical regions in $[0,1]^2$ which have $(1/3, 1/3)$ as one focus and the other focus is on the line segment $\overline{(1/3, 1/2), (\frac{1}{3}, \frac{2}{3})}$



$$\Rightarrow \bigcap \overline{\pi_1(D_\lambda)} = 1/3$$

$$\bigcap \pi_2(D_\lambda) = 1/3$$

and $(1/3, 1/3) \in \bigcap D_\lambda \subseteq \bigcap A_\lambda$.

constructing the larger set with the f.i.p. uses Zorn's Lemma.. (which is equivalent to the Axiom of Choice)

Def: $<$ is an order relation on W which satisfies

(0) $\forall a,b$ either $a<b$ or $b<a$

(1) $a<a$ never holds

(2) $a<b,\ b<c \Rightarrow a<c$

is called a simple or strict linear order.

If just satisfy (1) and (2), is a strict partial order.

DEF (1) If $(A, <)$ is a strict partial

$B \subseteq A$ is a subset. An **upper bound**

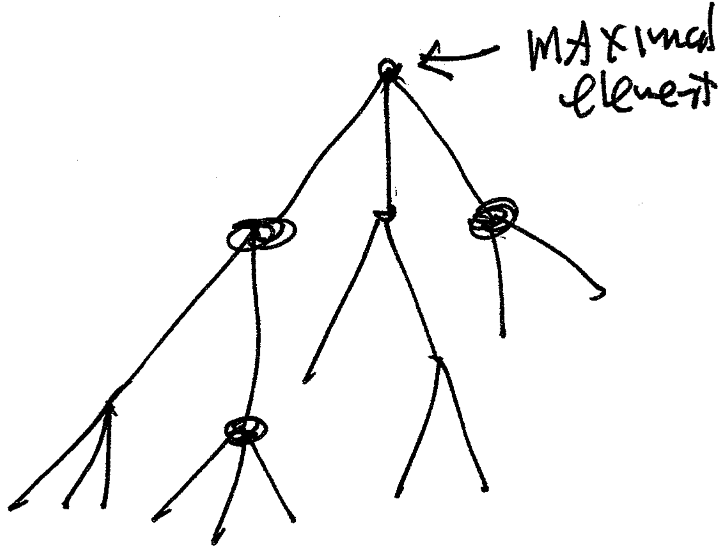
for B is some c with

$\forall b \in B$, either $c > b$ or $c = b$.

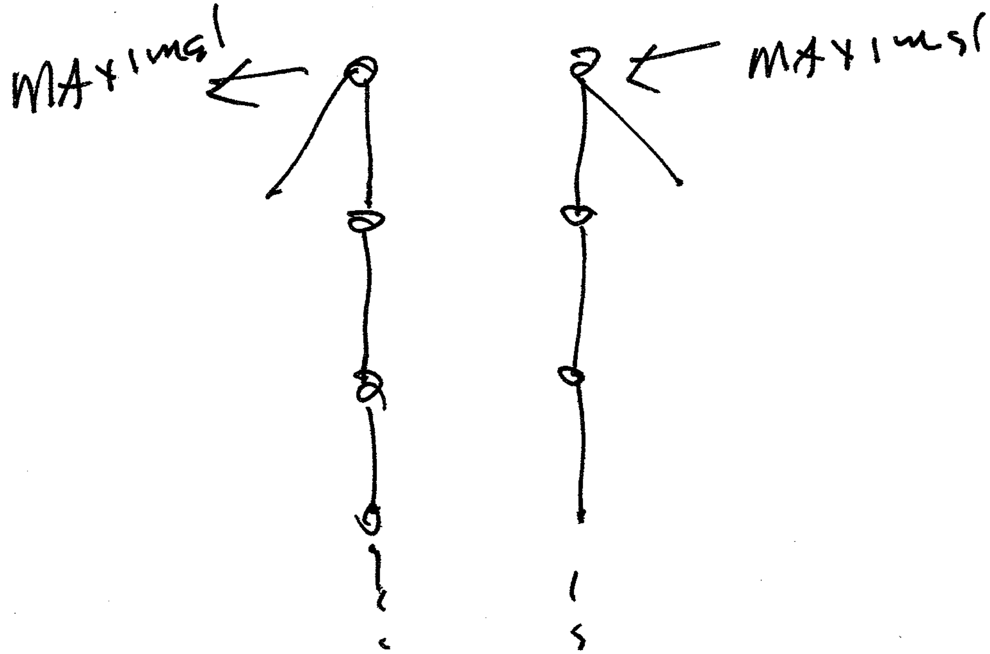
(2) $m \in A$ is a maximal element

If $a > m$ never happens.

or $\{a : a > m\} = \emptyset$.

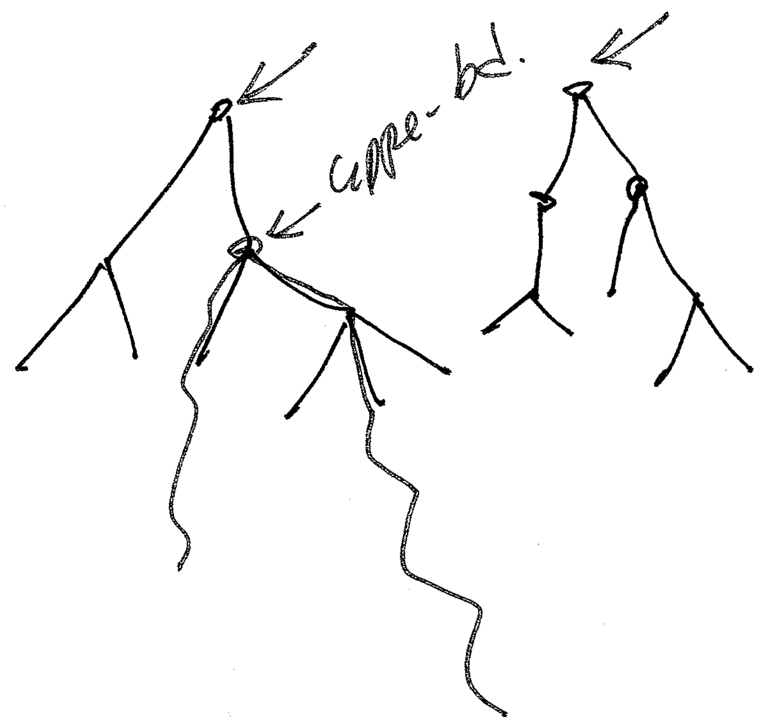


$a > b$ if it is higher in the tree and connected by branches.



Zorn's Lemma

$(A, <)$ is a strict partial order. If every simply ordered subset has an upper bound $\Rightarrow$ A has a maximal element.



The most common use ~~as~~ is to have A a collection of ~~set~~ subsets and

$$A_1 < A_2 \iff A_1 \subsetneq A_2$$