Tychonoff – Product of Cpt spaces is cpt.

~~Lemma~~ coss = collection of subsets.

Lemma 1: X is set $\mathcal{A}$ coss with f.i.p

$\Rightarrow \exists\, \mathcal{D}$ a coss which is maximal w.r.t. these two properties

(1) $\mathcal{A} \subseteq \mathcal{D}$

(2) $\mathcal{D}$ has the f.i.p.

Lemma 2: Assume $\mathcal{D}$ is maximal wrt fip

$\Rightarrow$ (a) $\mathcal{D}$ is closed under finite intersection

(b) If $A \cap D_\lambda \neq \emptyset \; \forall \lambda \; D_\lambda \in \mathcal{D}$

$\Rightarrow A \in \mathcal{D}$.

Proof 1, 2a last time

Proof of 2b: Assume A has the given properties

Let $\Sigma = \mathcal{D} \cup \{A\}$

Let $D_1, \ldots, D_n \in \Sigma$. If they are all in $\mathcal{D} \Rightarrow \cap D_i \neq \emptyset$ since $\mathcal{D}$ has the fip.

If one of them is A,

examine $D_1 \cap \ldots \cap D_{n-1} \cap A$

By part (a), $D_1 \cap \ldots \cap D_{n-1} := D \in \mathcal{D}$

$\Rightarrow$ by hypothesis $D \cap A \neq \emptyset$.

$\Rightarrow \Sigma$ has the f.i.p. and

So $\mathcal{D} = \Sigma$ or $A \in \mathcal{D}$. $\blacksquare$

$\boxed{\text{Proof of } T_o}$

Let $X = \prod_{\alpha \in \Lambda} X_\alpha$ with each X_α cpt

Let $\mathcal{A}$ be a coss of X with fip.

We show $\bigcap_{A \in \mathcal{A}} \bar{A} \neq \phi$ yielding cptness of X.

Use Lemma 1 to get $\mathcal{D}$ maximal wrt

(i) $\mathcal{A} \subseteq \mathcal{D}$

(ii) $\mathcal{D}$ has fip.

By (i), it suffices to get $\bigcap_{D \in \mathcal{D}} \bar{D} \neq \phi$.

Let $\pi_\lambda : X \to X_\lambda$ consider

$\{\pi_\lambda(D) : D \in \mathcal{D}\}$. is a coss X_λ and it has the fip since $\mathcal{D}$ has. Since X_λ is cpt, $\forall \lambda\ \exists$

$$x_\lambda \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\lambda(D)}$$

Let $\underline{x} = (x_\lambda)_{\lambda \in \Lambda}$. We show

$\underline{x} \in \overline{D}, \forall D \in \mathcal{D}$ so $\underline{x} \in \bigcap_{D \in \mathcal{D}} \overline{D}$

which is thus ~~em~~ nonempty.

Step 1: claim: If $\underline{x} \in \pi_\lambda^{-1}(U_\lambda)$ (U_λ open X_λ)

$\Rightarrow$ ~~$\pi_\lambda^{-1}(D_\lambda)$~~ $\pi_\lambda^{-1}(U_\lambda)$ intersects every element of $\mathcal{D}$.

Proof of claim: Pick $D \in \mathcal{D}$. Now

$X_\lambda \in \overline{\pi_\lambda(D)}$ by construction so

$\exists\ \underline{y} \in D$ with $U_\lambda \cap \pi_\lambda(D) = \pi_\lambda(\underline{y})$

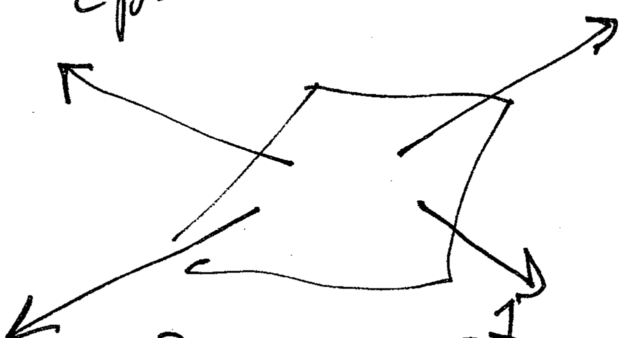
$\Rightarrow \underline{y} \in \pi_\lambda^{-1}(U_\lambda) \cap D$.

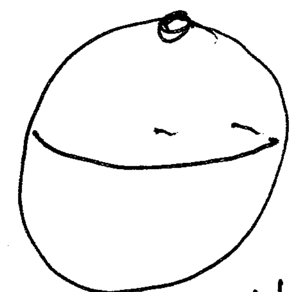
D Step 2: By Lemma 2 (b) This yields that every $\pi_\lambda^{-1}(U_\lambda)$ with $\underline{x} \in \pi_\lambda^{-1}(U_\lambda)$ is actually in $\mathcal{D}$. By Lemma 2(a) finite intersections of $\pi_\lambda^{-1}(U_\lambda)$ with this property are in $\mathcal{D}$. But these finite intersections are base elements for the product topology. Thus every base element of the prod. top. that contains $\underline{x}$ is in $\mathcal{D}$. But $\mathcal{D}$ has the f.i.p. Thus every base element that contains x intersects every element of $\mathcal{D}$. This means $\underline{x} \in \overline{D}$ $\forall D \in \mathcal{D}$. $\Rightarrow \underline{x} \in \bigcap_{D \in \mathcal{D}} D_\lambda$. ☒

e9//

$\mathbb{R}^2 \amalg \{\infty\}$ with correct topology yields S^2

non cpt　　　pt

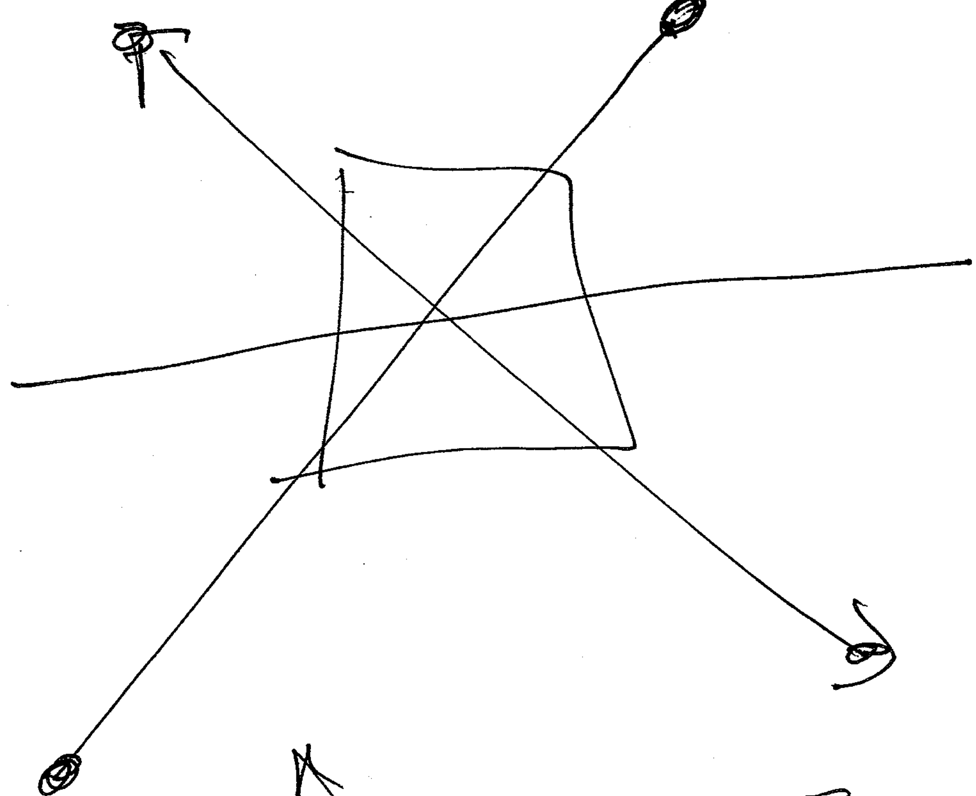




$\mathbb{R}^2 \amalg S^1$ with correct topology.

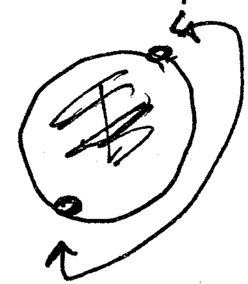
$\mathbb{R}^2 \cong \mathring{D}^2 = \{z : |z| < 1\}$

$D^2 = \{z : |z| \leq 1\}$.

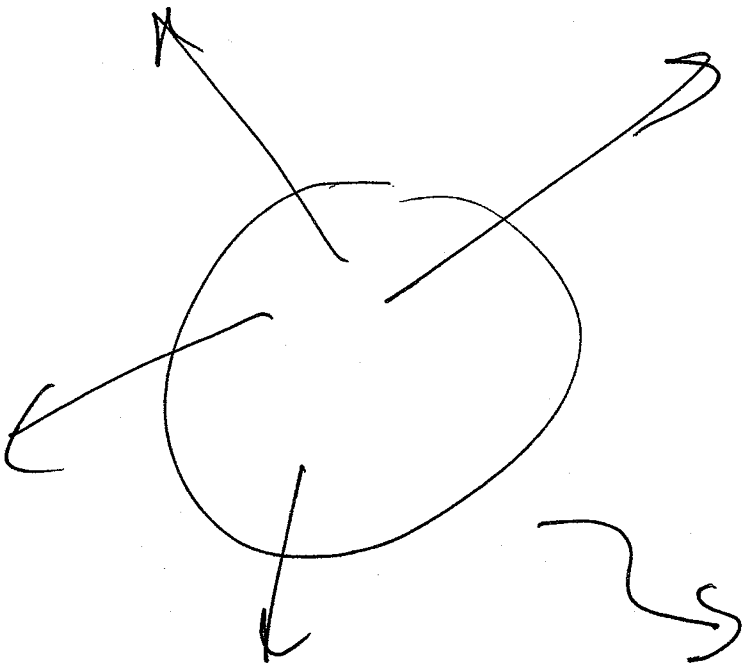


~~direction~~ angle

each line yields
a point at
infinity



$\mathbb{R}P$



each directed line
is a point at ∞

$\sim D^2$

3 ways to "compactify the plane"

DEF: (a) A compactification of a space X is a compact HD space Y with

$$X \subseteq Y$$

and $\overline{X} = Y$

So $Y = X \amalg$ other stuff with the correct topology.

(b) Two compactifications Y_1 and Y_2 are equivalent if $\exists$ homeomorphism $h : Y_1 \to Y_2$ with $h|_X = \mathrm{id}_X$

DEF: $h : X \to Z$ is an embedding (imbedding) of X into Z is h is a homeomorphism onto its image.

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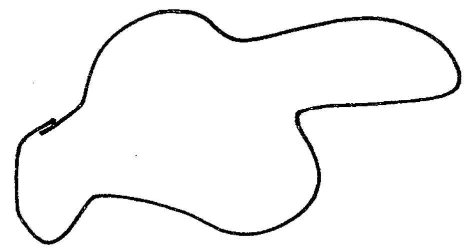
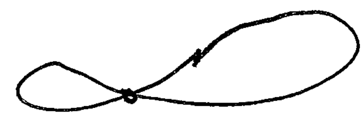
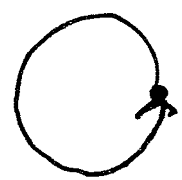
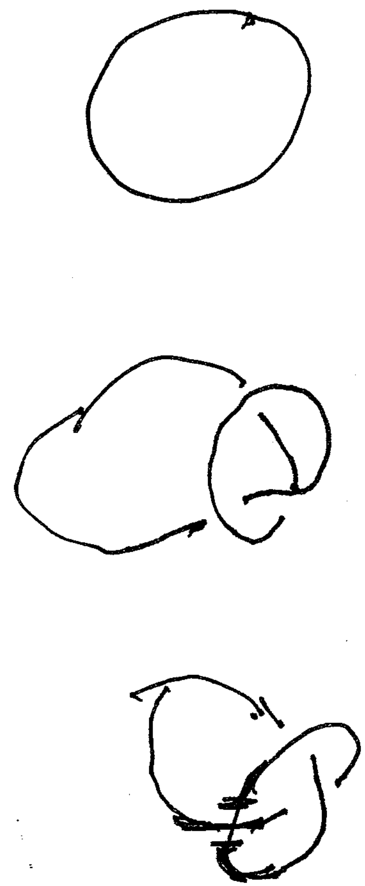
eg//

$S^1 \longrightarrow$  embedding

$S^1 \longrightarrow$  not an embed. (not injective)

$[0,1) \xrightarrow{e^{2\pi i t}}$  continuous, injective not an embed.

$S^1 \longrightarrow \mathbb{R}^3$



embeddg.

non equivalent embeddings

knots.

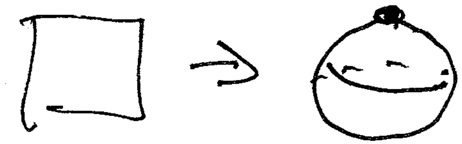
<u>Lemma</u> Say $h : X \to Z$ is an embedding of X into the cpt HD space Z

$\Rightarrow$ $\exists$ a compactification Y of X ~~and~~ and an embedding $H : Y \to Z$ with equals h on X.

e.g.// $h : \mathbb{R}^2 \to S^2$ via inverse of stereographic projection. This yields a compactification of $\mathbb{R}^2$ homeomorphic to S^2

<u>Proof</u> Let $X_0 = h(X) \subseteq Z$.

$Y_0 = \overline{X_0} \subseteq Z$



Then Y_0 is a compactification of X_0. [closed subsets of cpt HD are cpt].

Next time — pull this back to X