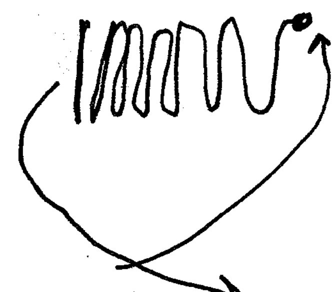
DEF: X is a topological space. A compactification Y of X is a compact H.D. with $X \subseteq Y$ and $\overline{X} = Y$.

Remark A space has many compactifications usually. The smallest is the one point so $Y = X \cup \{pt\}$. The largest is Stone-Čech which we do today.

Natural way to obtain a compactification is to embed X in a cpt HD Z as in the next lemma.

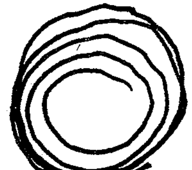
Lemma $h : X \to Z$ is an embedding into Z a cpt HD. $\Rightarrow \exists$ a compactification Y of X and an embedd $H : Y \to Z$ with $H_{|X} = h$.

Examples (a) $h : (0,1) \to [-1,1]^2$

$h(x) = (x, \sin \frac{1}{x})$ 

is an embedding

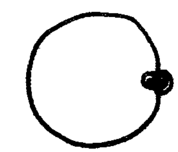
$\overline{h(0,1)}$ is $h(0,1) \cup pt \cup$ line segment

(b)  $\overline{h(0,1)} =$ Spiral $\cup$ circle

(c) $h(x) = x$ $\quad h: (0,1) \to \mathbb{R}$

and $\overline{h(0,1)}$ is $[0,1]$

(d) $h(t) = e^{2\pi i t}$



$\overline{h(0,1)} = S^1$.

<u>Proof</u> $\overline{X}_0 = h(X) \subseteq Z$

$Y_0 = \overline{h(X)} \subseteq Z$

So Y_0 is a cptification of X_0.

Let A be an abstract set $A \cap X = \emptyset$

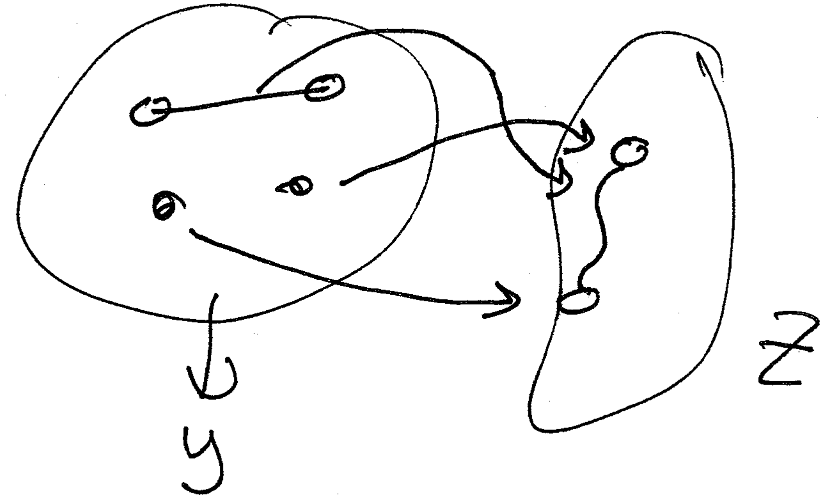
and there is a bijection

$k: A \to Y_0 - X_0$

Let $Y = \cancel{X \cup X}\ X \cup A$

~~with the topology~~.

$H : Y \to Y_0$



$H(x) = h(x) \quad x \in X$

$H(a) = K(a) \quad a \in A$

and topologize Y so that U is open in Y $\Longleftrightarrow$ $H(U)$ is open in Z

Since H is a bijection and a homeomorphism, ie an embedding.

This compactification is called the one induced by h.

Big question

$f : X \to W$ is continuous

Y is a compactification of X

does there exist an extension

$\bar{f} : Y \to W$ continuous.

$$\bar{f}|_X = f \iff \begin{tikzcd} X \arrow[r, "f"] \arrow[d, phantom, sloped, "\subseteq"] & W \\ Z \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

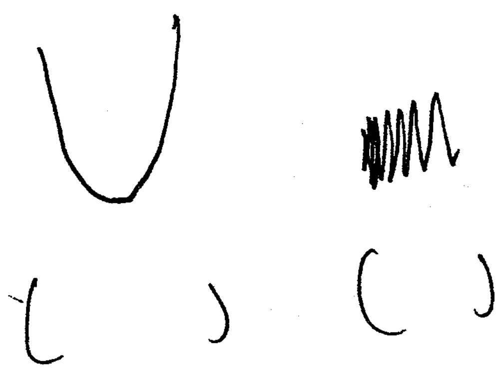
__examples__

$h: (0,1) \to [-1,2]$

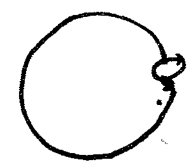
$\cancel{f(x)}$ $h(x) = x$

$f: (0,1) \to \mathbb{R} = W$ cont. does it exten

extends $\iff$ $\left.\begin{array}{l} \lim_{x \to 0^+} f(x) \text{ exists} \\ \lim_{x \to 1^-} f(x) \text{ exists.} \end{array}\right]$



(b) $h : (0,1) \to [-2,2]^2$

$h(t) = e^{2\pi i t}$  $f : (0,1) \to \mathbb{R}$

extends $\Leftrightarrow$ two limits above exist <u>and</u> are equal

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Given a space X, a Stone-Čech cptification is a space βX with the propery that

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{R} \\ \cap & & \uparrow \bar{f} \\ & & \beta(X) \end{array}$$

f continuous and bounded.

extends to $\bar{f} : \beta(X) \to \mathbb{R}$

<u>$\forall f$</u>.

$$\begin{tikzcd} X \arrow[dr, "f"'] & \subseteq & \beta(X) \arrow[d, "\bar{f}"] \\ & & \mathbb{R} \end{tikzcd}$$

or any cpt HD space we will see

Why f bdd?

Well, $\beta(X)$ is cpt so $\bar{f}(\beta X)$ is bounded so if the extension exist $f(X)$ must be bounded.

βX is the biggest compactification.

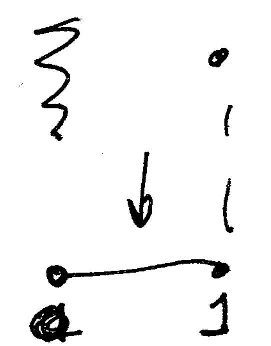
Theorem βX exists $\Leftrightarrow$ X is HD and completely regular.

Reminders

X is completely regular if it is T_1 (pts are closed), if pts and closed sets can be separated by cont funct into $[0,1]$

i.e. $x_0 \notin A$, A closed $\Rightarrow$ $\exists$ cont $f: X \to [0,1]$ with $f(x_0) = \{1\}$

$f(A) = \{0\}$



~~regular $\Rightarrow$ completely reg. $\Rightarrow$ normal~~

normal $\Rightarrow$ C.R. $\Rightarrow$ Reg.

(2) Subspace of C.R. is C.R.

(3) Cpt HD is normal $\Rightarrow$ C.R.

(4) If X has a cptification X is homeomorph to a subspace of C.R $\Rightarrow$ X is C.R.

Theorem (existence of Stone-Čech)

Assume X is C.R. $\Rightarrow \exists$ a cptification Y of X so that every bounded cont function $f: X \to \mathbb{R}$ extends uniquely to a continuous $\bar{f}: Y \to \mathbb{R}$.

Proof Let $\{f_\lambda\}_{\lambda \in \Lambda}$ be the collection of all **bounded** real valued functions on X.

Let $I_\lambda = [\inf f_\lambda(X), \sup f_\lambda(X)]$

and $Z = \prod_{\lambda \in \Lambda} I_\lambda$ is cpt by Tychonoff.

define $h : X \to \prod_{\lambda \in \Lambda} I_\lambda$ by

$$(h(x))_\lambda = f_\lambda(x) \quad \forall \lambda \in \Lambda$$

or $h(x) = (f_\lambda(x))_{\lambda \in \Lambda}$

We need

(1) h is an <u>embedding</u> (next time).

Then let Y be the compactification induced by h. So $\exists$ ~~expu~~ embed by previous lemma $H : Y \to \prod I_\lambda = Z$

Such that $H = h$ on $X \subseteq Y$.

(2) Check extension property.

Note that that any bound cont $f : X \to \mathbb{R}$ is one of the f_λ, say f_{λ_0}

Claim is $\Pi_{\lambda_0} \circ H : Y \to I_{\lambda_0}$ is the desired extension since $x \in X$

$$\Pi_{\lambda_0} \circ H(x) = \Pi_{\lambda_0} h(x)$$
$$= \Pi_{\lambda_0}\left((f_\lambda(x))_{\lambda \in \Lambda} \right)$$
$$= f_{\lambda_0}(x).$$

(3) uniqueness of the extension next time ∎

Also next time, βX is unique up to homeomorphism.