Theorem: X is completely regular $\Rightarrow$ $\exists$ a compactification Y with the property that any bounded continuous $f: X \to \mathbb{R}$

~~ha~~ can be uniquely extended to $\bar{f}: Y \to \mathbb{R}$

$$\begin{tikzcd} X \arrow[r, phantom, "\subseteq"] \arrow[dr, "f"'] & Z \arrow[d, "\bar{f}"] \\ & \mathbb{R} \end{tikzcd}$$

Terminology: This compactification is called the concrete Stone-Čech compactification.

Ideas in proof from last time:

$\{f_\lambda\}_{\lambda \in \Lambda}$ all bdd, cont, $\mathbb{R}$-valued functions

$I_\lambda = [\inf f_\lambda(X), \sup f_\lambda(X)] \quad \forall \lambda \in \Lambda.$

$\prod I_\lambda$ is compact by Tychonoff.

Define $H : X \to \prod I_\lambda$ via

$$(H(x))_\lambda = f_\lambda(x)$$

$\to$ (1) This is an embedding (to be proved)

let Y be the induced compactification

(2) Any $f : X \to \mathbb{R}$ cont, bdd is some f_{λ_0}

$\Rightarrow \pi_{\lambda_0} \circ H$ extends f_{λ_0}

$\to$ (3) uniqueness of extension (to be proved)

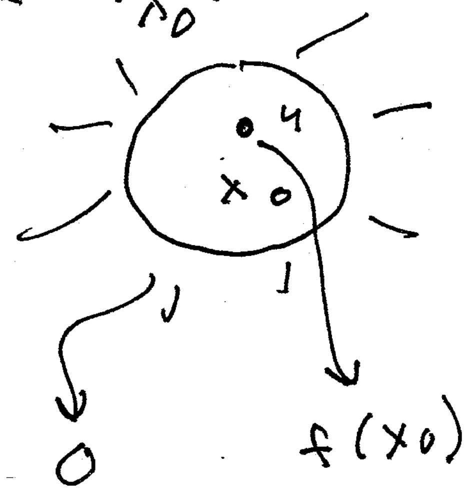
Lemma to prove (1): X is HD and assume

$\exists \{f_\lambda\}_{\lambda \in \Lambda}$ of cont $f_\lambda : X \to \mathbb{R}$

with the property that for any $x_0 \in X$ $\exists$ index λ_0

and open nhd U of x_0

with $f_{\lambda_0}(x_0) > 0$ and $f_{\lambda_0}(U^c) = 0$



$\Rightarrow F : X \to \mathbb{R}^\Lambda$ defined by

$(F(x))_\lambda = f_\lambda(x)$ is an embedding.

Proof

(a) continuous since $\mathbb{R}^{\Lambda}$ has the product topology

(b) injective: If $x \neq y$ by HD $\exists$ disjoint nbhds U and V of x and y, respectively. By hypothesis $\exists\, i_0$ with $f_{i_0}(x) > 0$ $f_{i_0}(U^c) = 0$

but $y \in U^c \Rightarrow f_{i_0}(y) = 0$

So $f_{i_0}(x) \neq f_{i_0}(y)$.

U $\quad V$ $\cdot y$

(c) F^{-1} is cont. or U is open in X

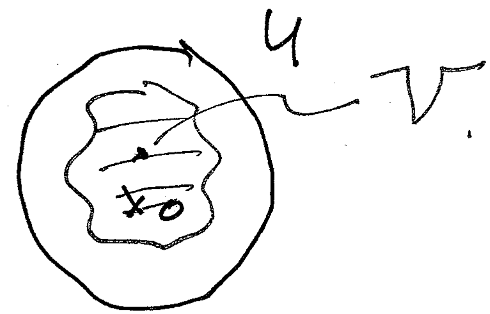
$\Rightarrow$ $F(U)$ is open in $Z := F(X)$.

So say $z_0 \in F(U)$ we must find an open W with $z_0 \subseteq W \subseteq F(U)$.

Let x_0 be such that $F(x_0) = z_0$.

Pick an index λ_0 with $\cancel{f(\lambda_0}$ $f_{\lambda_0}(x_0) > 0$

$f_{\lambda_0}(U^c) = 0$



Let $V = \Pi_{\lambda_0}^{-1}((0,\infty))$

which is open in ~~X~~ Z by cont of Π_{λ_0}.

and $W = V \cap Z$ is open in Z.

$z_0 \in V$ since $\Pi_{\lambda_0}(z_0) = \Pi_{\lambda_0}(F(x_0))$

$= f_{\lambda_0}(x_0) > 0.$

Also $W \subseteq F(U)$ ~~since~~ proof: pick

$z \in W \Rightarrow z = F(x)$ some $x \in X$

and $\Pi_{\lambda_0}(z) \in (0, \infty)$. Since

$0 < \Pi_{\lambda_0}(z) = \Pi_{\lambda_0} F(x) = f_{\lambda_0}(x)$

and f_{λ_0} vanishes outside U,

So x must be in U and so

$z = F(x)$ is in $F(U)$.

$\Rightarrow W \subseteq F(U)$ ☒

COR: $\mathbb{X}$ is completely regular $\iff$ it embeds in $[0,1]^{\Lambda}$ for some Λ.

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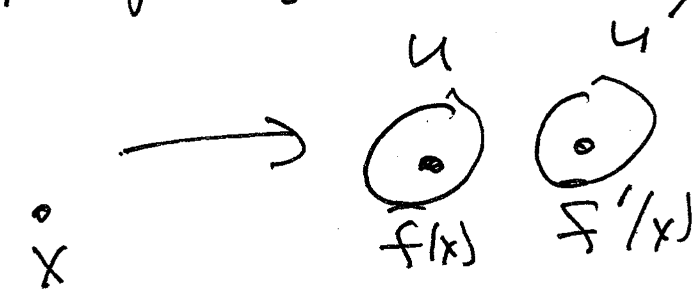
(3) Uniqueness of extension: $A \subseteq \mathbb{X}$ and $f: A \to Z$ continuous into a HD space $Z \Rightarrow$ if there is a continuous extension $\bar{f}: \bar{A} \to Z$ then it is unique.

__Proof__ Let $\bar{f}$ and $\bar{f}'$ be different extensions

So $\exists\, x \in \bar{A}$ with $\bar{f}(x) \neq \bar{f}'(x)$. By

HD let U and U' be disjoint ~~nbd~~ neighborhoods

of $\bar{f}(x)$ and $\bar{f}'(x)$ respectively.



By continuity, $\exists$ nbd W of x with $\bar{f}(W) \subseteq U$ W' of x with $\bar{f}'(W') \subseteq U'$

Let $V = W \cap W'$ is open and hits

$\bar{A} \Rightarrow \exists$ pt $y \in A \cap V$

$\Rightarrow \bar{f}(y) = \bar{f}'(y)$ but $\bar{f}(y) \in U$
$\bar{f}'(y) \in U'$

Th'm Let $\underline{X}$ be completely regular and $\mathcal{Y}$ be its concrete Stone-Čech compactification. It has the following property.

Given any continuous map $f : \underline{X} \to C$ with C compact, HD $\Rightarrow \exists$ unique cont. extension $\bar{f} : \mathcal{Y} \to C$

$$\begin{tikzcd} \underline{X} \arrow[r, hook] \arrow[dr, "f"'] & \mathcal{Y} \arrow[d, dashed, "\bar{f}"] \\ & C \end{tikzcd}$$

Proof: C is cpt HD $\Rightarrow$ normal $\Rightarrow$ completely regular so by the cor above $\exists$ embedding $C \hookrightarrow [0,1]^{\Lambda}$ for some Λ.

So we assume $C \subseteq [0,1]^{\Lambda}$.

So $f: X \to C$ now has component function $f_\lambda: X \to [0,1] \subseteq \mathbb{R}$. By concrete Stone-Čech construction, this extends to $\bar{f}_\lambda: Y \to [0,1]$. Let $F: Y \to \mathbb{R}^{\Lambda}$ via

$$(F(x))_\lambda = \bar{f}_\lambda(x)$$

So F is cont since $\mathbb{R}^{\Omega}$ has the product topology. We need $F(y) \subseteq C$.

Proof of this,

$$F(y) = F(\overline{\underline{X}}) \underset{\text{by continuity.}}{\subseteq} \overline{F(\underline{X})} = \overline{f(\underline{X})} \subseteq \overline{C} \underset{C \text{ is cpt}}{=} C$$

DEF: A diagramatic Stone-Čech compactification is a cpt HD space Y with an embedding $\varphi: X \to Y$ with $\overline{\varphi(X)} = Y$ and $\forall$ cpt HD C, cont $f: X \to C$ cont $\Rightarrow \exists!$ extension

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\bar{f}"] \\ & C \end{tikzcd}$$

<u>**Lemma**</u> If Y and Y' are s~~tone-Čo~~ diagramatic Stone-Čech compactification of a completely regular X under φ and φ'

$\Rightarrow$ $\exists$ a homeomorphism $h: Y \to Y'$ with

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & Y \arrow[d, "h"] \\ & Y' \end{tikzcd}$$

<u>**Proof**</u>: HW

Remark: It is common to speak of the Stone-Čech compactification when really one is talking about a homeomorphism class.