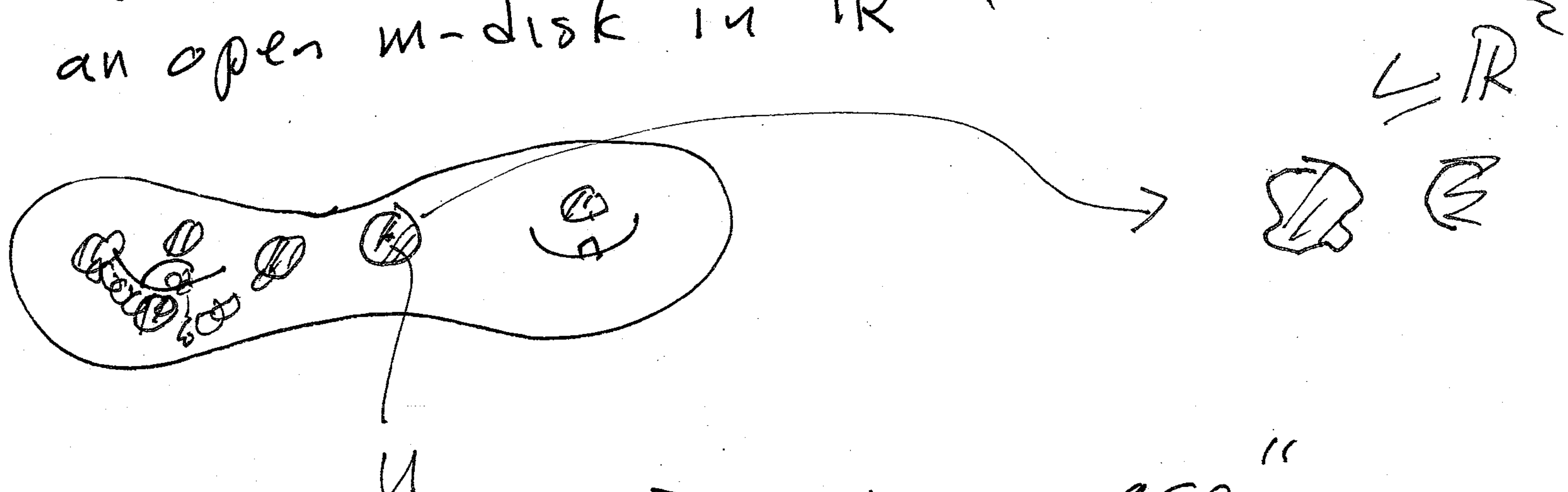
Chapt 36 – embedding manifolds in $\mathbb{R}^n$

DEF: An m-manifold is a HD space with a countable base. Such that every point has a neighborhood that is homeomorphic to an open m-disk in $\mathbb{R}^m$.



"Locally looks like Euclidean space"

eg// S^m is a m-dim mfd. $\forall m > 1$

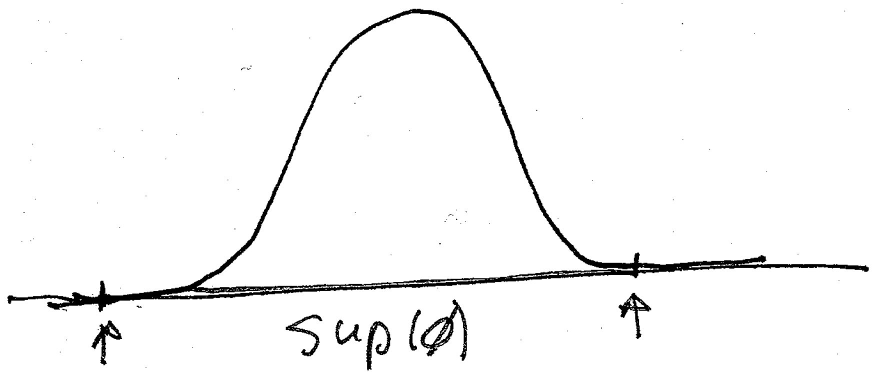
- This is a topological or C^0-manifold
- commonly smooth or C^r-manifold's are considered where you can do calculus ~~...~~ (Differential Topology and Diff Geom).

__Theorem__: Any __compact__ mfd can be embedded in $\mathbb{R}^N$ for some N.

__DEF__: $\phi: X \to \mathbb{R}$, its support

$$\operatorname{Supp}(\phi) = \overline{\phi^{-1}(\mathbb{R} - \{0\})}$$

So x is __not__ in support if it has a nbd U with $\phi(U) = 0$

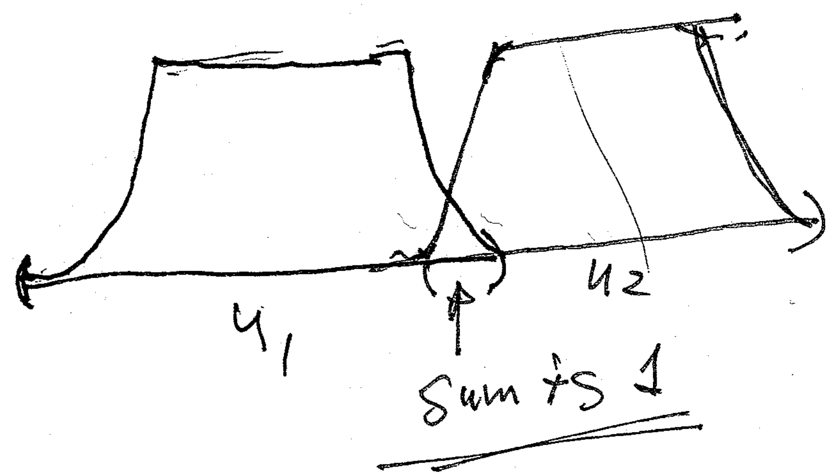


graph $\phi : \mathbb{R} \to \mathbb{R}$

DEF: $\{U_1, \dots, U_n\}$ is an open cover of X, a family of functions $\phi_i : X \to [0,1]$, $i = 1, \dots, n$, is a partition of unity ~~sup~~ subordinated or dominated to be cover if

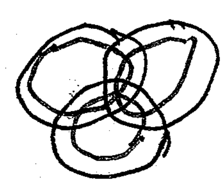
(1) $\sup(\phi_i) \subseteq U_i \quad \forall i$

(2) $\sum_{i=1}^{n} \phi_i(x) = 1 \quad \forall x \in X$.



Lemma: Let $\{U_1, \ldots, U_n\}$ be a finite open cover of a normal space X $\Rightarrow$ $\exists$ p.o.u subordinate to the cover.

Proof Step 1: Claim: $\exists \{V_1, \ldots, V_n\}$ open still covers and $\overline{V_i} \subseteq U_i \ \forall i$



(Shrinking Lemma)

Proof of claim By induction.

$A = X - (U_2 \cup \cdots \cup U_n)$, is closed

Since $\{U_i\}$ cover $\Rightarrow$ $A \subseteq U_1$. By

Normality $\exists$ V_1 open with

$A \subseteq \overline{V_1} \subseteq U_1$ and $\{V_1, U_2, \ldots, U_n\}$

still covers

Inductive step: Given $\{V_1, \ldots, V_{k-1}, U_k, \ldots, U_n\}$

covers. $A = X - \{V_1 \cup \cdots \cup V_{k-1}\} - \{U_{k+1}, \ldots, U_n\}$

$\Rightarrow$ $A \subseteq U_k$ by normality find V_k

with $A \subseteq \overline{V_k} \subseteq U_k$ and

$\{V_1, \ldots, V_{k-1}, V_k, U_{k+1}, \ldots, U_n\}$ still covers.

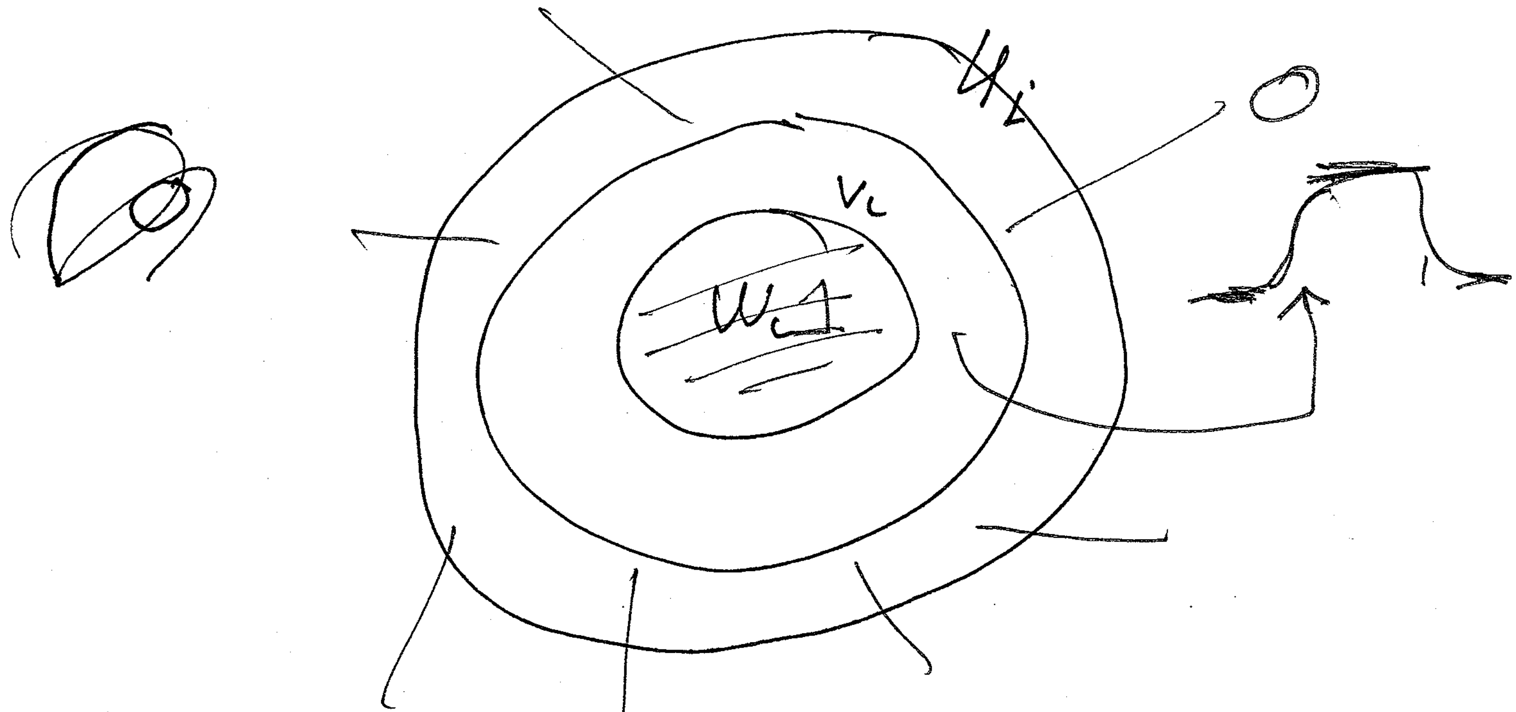
completing the induction.

<u>Step 2</u> Proof of theorem. Given $\{U_i\}_{i=1}^n$ and construct $\{V_i\}_{i=1}^n$ as in step 1 and ~~with~~ which do it again to get $\{W_i\}_{i=1}^n$ covers and $\overline{W_i} \subseteq V_i$.

By Urysohn's lemma, for each i

$\exists\ \psi_i : X \to [0,1]$ $\psi_i(\overline{W_i}) = \{1\}$

$\psi_i(V_i^c) = \{0\}$



Now $\operatorname{sup}(\psi_i) \subseteq \overline{V_i} \subseteq U_i$

Since $\{W_i\}$ covers $\forall i$ $\sum_{i=1}^{n} \psi_i(x) > 0$

$\forall x$, let $\psi(x) = \sum_{i=1}^{n} \psi_i(x)$ and

$\forall j$ $\phi_j(x) = \dfrac{\psi_j(x)}{\psi(x)}$ is the p.o.u.

Proof of main theorem: Using compactness

pick a finite cover $\{U_1, .., U_n\}$ where

~~with~~ each U_i can be embedded in

$\mathbb{R}^m$ say by $g_i : U_i \to \mathbb{R}^m$.

Since X is cpt HD $\Rightarrow$ normal $\Rightarrow$

$\exists$ p.o.u. ϕ_i subordinate to $\{U_i\}$.

Let $A_i = \mathrm{sup}(\phi_i)$ define $h_i : X \to \mathbb{R}^m$

via
$$h_i(x) = \begin{cases} \phi_i(x) \cdot g_i(x) & x \in U_i \\ \vec{0} & x \in X - A_i \end{cases}$$

continuous by patching lemma. (agree on the overlap of domains of def.)

$$F : X \to \underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{n \text{ times}} \times \underbrace{\mathbb{R}^m\times\cdots\times\mathbb{R}^m}_{n\text{-times}}$$

$$F(x) = (\phi_1(x), \ldots, \phi_n(x), h_1(x), \ldots, h_n(x))$$

continuous since each component is. Since X is compact, F is an embedding if we can prove it is injective.

So assume $F(x) = F(y)$

or $\phi_i(x) = \phi_i(y)$ and $h_i(x) = h_i(y)$ $\forall i$

Now $\phi_i(x) > 0$ for some i since $\sum \phi_i(x) = 1$. For the same i, $\phi_i(y) = \phi_i(x) > 0$

But for this i

$$\phi_i(x)\, g_i(x) = h_i(x) = h_i(y) = \phi_i(y)\, g_i(y)$$

but $\phi_i(x) = \phi_i(y) > 0$

$$\Rightarrow \quad g_i(x) = g_i(y)$$

But g_i was an embedd. so $x = y$. $\blacksquare$

Optimum Result: When M is smooth
Whitney showed $M \subseteq \mathbb{R}^{2m}$ smoothly.