Topological Analysis of data

① Advantages:

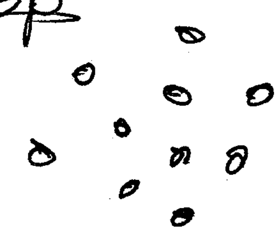
(a) coordinate free

(b) metric free

(c) robust to noise and ~~sm~~ perturbations

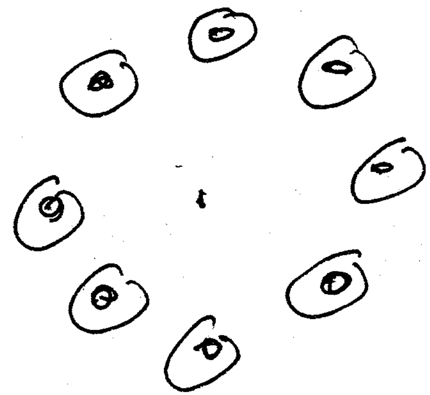
② Disadvantages:

(a) Data is usually a collection of disjoint points ⇒ ~~notop~~ no topology



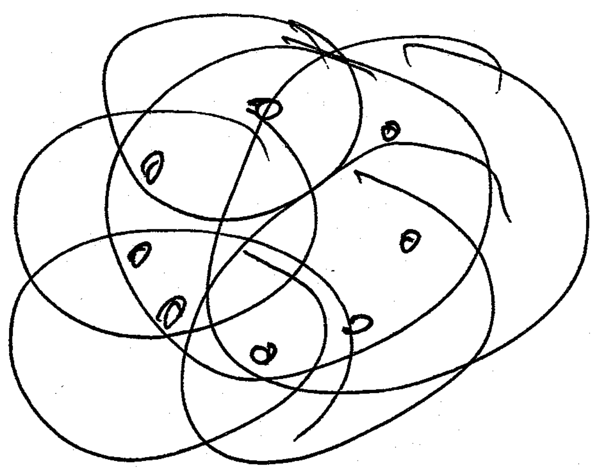
(b) computationally expensive.

Getting around 2(a).



→ fatten points to disks

Too small disks ⇒ no info



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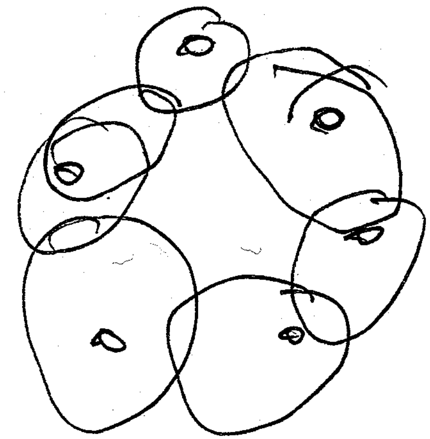


right size yields information.

How do you know correct size a priori?

Soln: keep track of all scales.

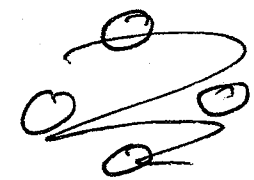
Parameterized topology or persistent topology.

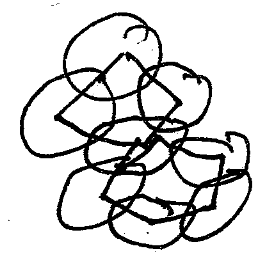
Tells you the topology at various scales.

Q How do we compute "the ~~topogol~~ topology"

We need numbers or algebraic objects.

eg # of "holes"



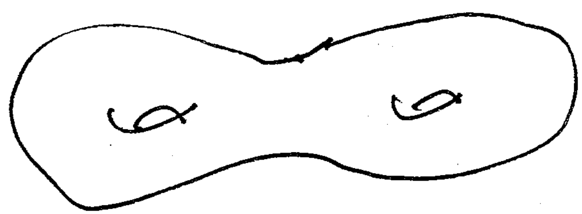


COULD USE π_1 (fattened data)

nonabelian group!

prefer a vector space or a number.

This is provided by the homology with field coefficients. (field $= \mathbb{R}$)

 $\longrightarrow H_1(M; \mathbb{R}) \cong \mathbb{R}^4$

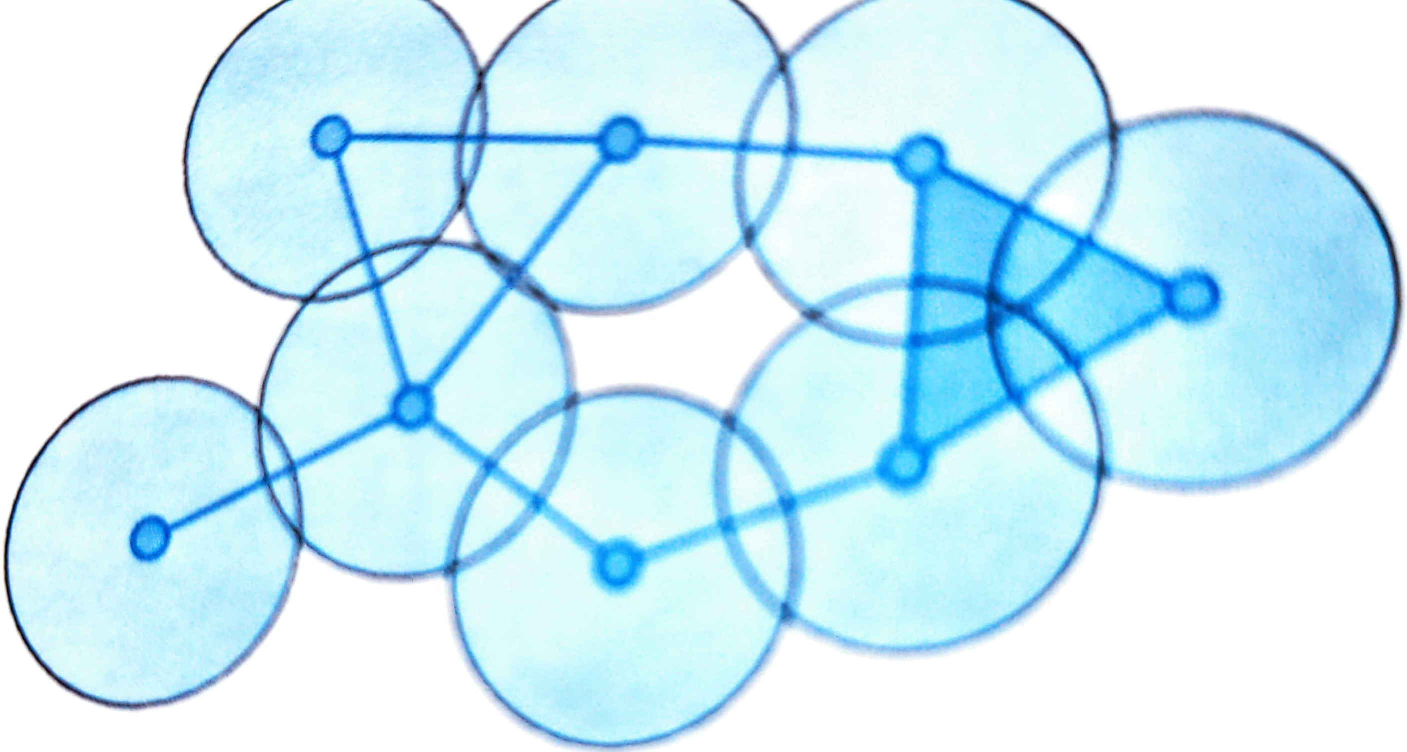
even simpler beti # $1 = 4$

$= \text{rank}(H_1)$

How do we compute this?

We need a combinatorial object ~~to re~~ that captures the essential topology.

- Idea: Turn the fattened space into simplicial complex. called Čech-complex or Rips-Vieroris, ..or...



ech complex is a combinatorial approxin

Notice - the simplicial complex has the same homotopy type as the fattened space

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Finally, Applications of TDA need to be "functorial".

Topics to study

(1) Category theory
(2) Simplicial complex
(3) Simplicial homology
(4) persistent homology and computations.

Category Theory (Quick Trip)

DEF of a category $\mathcal{C}$.

(1) Collection or class of objects $ob(\mathcal{C})$.

(2) collection or class of arrows or morphisms $hom(\mathcal{C})$ with

(a) each arrow f has a source A and target B and is denoted

$$f : A \to B$$

could be empty ← $hom(A,B)$ = all arrows A to B

(b) A distinguished arrow $1_A \in hom(A,A)$

(3) and a composition rule.

$$\hom(B,C) \times \hom(A,B) \longrightarrow \hom(A,C)$$

written $g \circ f$: $A \to C$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

These satisfy

(4) $f \circ (g \circ h) = (f \circ g) \circ h$.

(5) $f : A \to B \qquad g : B \to C$

$1_B \circ f = f \qquad g \circ 1_B = g$.

__Often__: an arrow is a structure preserving map, but not always

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__DEF__: $f: A \to B$ is an __equivalence__ or __isomorphism__ if $\exists\, g: B \to A$ with $g \circ f = 1_A$, $f \circ g = 1_B$.

Examples

(1) SET objects = all sets
arrows = functions.

(2) Grp objects = all groups
arrows = homomorphisms

(3) Vect($\mathbb{R}$) all finite dim v.s. with linear transformations.

(4) Top   objects = all topological spaces
arrows = continuous function
Isomorphism = homeomorphism.

(5) All partially ordered sets
and arrows! $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$

Categories ~~without~~ of different sort.

(6) Let S be a single partially ordered set. The objects are the elements of S and there is an arrow $A \to B \Leftrightarrow A \le B$.

objects are not sets and arrows are
not structure presey morphisms

Only isomorphism is 1_A.
$(A \leq B \quad B \leq \cancel{A} \Rightarrow A = B)$

(7) Let G be a group and the
single object in the category.
arrows are elements of G.
$\hom(G, G)$
composition rule is the group product.

every arrow is an isomorphism
since $\exists a^{-1} \quad aa^{-1} = id$.