Top 2.3

$P(x_0, x_1)$ = all paths $x_0 \to x_1$

$*: P(x_0,x_1) \times P(x_1,x_2) \longrightarrow P(x_0,x_2)$

$f \underset{p}{\simeq} f'$ is homotopic rel endpoints

equiv on $P(x_0,x_1)$, classes $[f]$

$\overline{P}(x_0,x_1)$ = all equiv classes.

$*: \overline{P}(x_0,x_1) \times \overline{P}(x_1,x_2) \longrightarrow \overline{P}(x_0,x_2)$

$[f * f'] = [f] * [f']$

is welldefined.

DEF: $e_x : [0,1] \to X \quad e_x(s) = x \quad \forall s \in [0,1]$

$f : [0,1] \to X, \quad f^{-1}(s) = f(1-s)$

Th'm: (1) $[f] * ([g] * [h]) = ([f] * [g]) * [h]$

(2) $[f] * [e_{x_1}] = [f]$

$[e_{x_0}] * [f] = [f]$

(3) $[f] * [f^{-1}] = [e_{x_0}]$

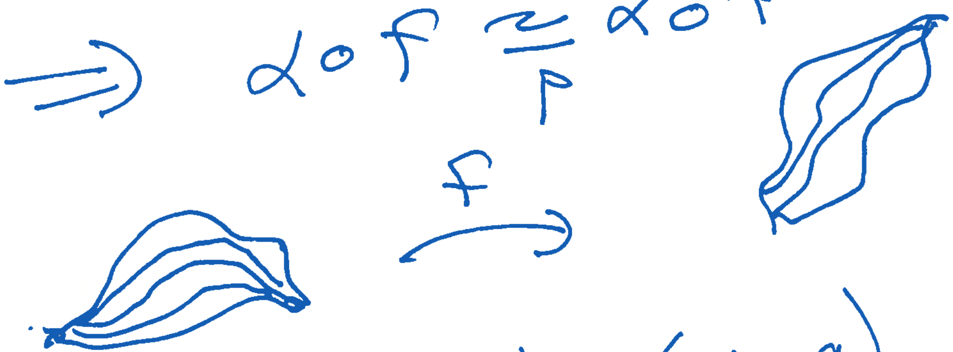
$[f^{-1}] * [f] = [e_{x_1}]$

## Prelim

A $f, f' : I \to X$

$\alpha : X \to Y$ cont

$f \simeq f' \Rightarrow \alpha \circ f \underset{p}{\simeq} \alpha \circ f'$




B $\alpha \circ (f * g) = (\alpha \circ f) * (\alpha \circ g)$


or $\alpha_*([f] * [g]) = \alpha_*[f] * \alpha_*[g]$

$\alpha_*[f] = [\alpha \circ f]$

(c) $f: I \to X$ path

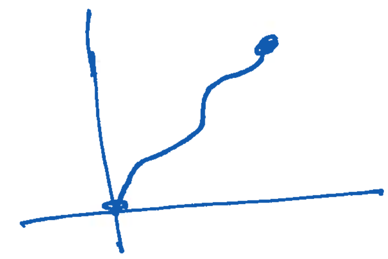
$\varphi: I \to I$

$\varphi(0) = 0 \quad \varphi(1) = 1$

and is homeomorphism

is called a reparameterization

and it yield $f \circ \varphi$ (time change)



<u>FACT</u> If $\varphi$ is a reparameterization

then $f \circ \varphi \underset{p}{\simeq} f$ or $[f \circ \varphi] = [f]$

Proof d(c): $\varphi$ and id are both paths $0 \to 1$ so since $I$ convex

$$\varphi \underset{p}{\simeq} id$$

$\Rightarrow$ (A) $\quad f \circ \varphi \underset{p}{\simeq} f \circ id = f.$

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Proof of Theorem – (2) last time

(3) now. define $L^{-1}(s) = 1 - s$

$L * L^{-1}$ is a path $0 \to 0$

& $e_0$ " " " $0 \to 0$

In $I$ convex

$$L * L^{-1} \underset{P}{\simeq} e_0$$

$$\Rightarrow \quad f \circ (L * L^{-1}) \underset{P}{\simeq} f \circ e_0 \quad \text{by } A$$

$$\Rightarrow \quad (f \circ L) * (f \circ L^{-1}) \underset{P}{\simeq} f \circ e_0$$

$$f * f^{-1} \underset{P}{\simeq} e_{x_0}$$

Proof 3

$f * (g * h) = \alpha$

does $f$ for $s \in [0, 1/2]$

$g$ " $s \in [1/2, 3/4]$

$h$ " $s \in [3/4, 1]$

$(f * g) * h = \beta$.

does $f$ for $s \in [0, 1/4]$

$g$ for $s \in [1/4, 1/2]$

$h$ for $s \in [1/2, 1]$.

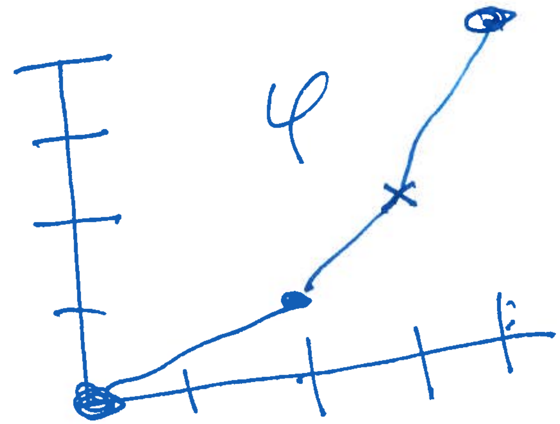
Define piecewise linear $\varphi : [0,1]$.

with $\varphi(0) = 0$

$\varphi(1/2) = 1/4$

$\varphi(3/4) = 1/2$

$\varphi(1) = 1$




$$\alpha = \beta \circ \varphi$$

$$\Rightarrow \alpha \underset{p}{\simeq} \beta.$$

## Change of base point.


$\alpha$ is a path $x_0 \to x_1$. Define

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

via $\quad \hat{\alpha}([f]) = [\alpha^{-1}] * [f] * [\alpha]$

$$= [\alpha^{-1} * f * \alpha]$$

DEF: $\varphi: G \to H$ map of groups is a homomorphism if

$$\varphi(g_1 g_2) = \varphi(g_1)\,\varphi(g_2)$$

and an isomorphism if

$\varphi$ is bijective and

$\varphi^{-1}$ is also a homomorphism.

functional inverse

Let $\beta = \alpha^{-1}$ : $\hat{\beta} : \pi_1(X, x_1) \to \pi_1(X, x_0)$

is a homomorphism by the same argument

$$\hat{\beta}([h]) = [\alpha] * [h] * [\alpha^{-1}]$$

Check $\hat{\alpha}\hat{\beta} = \mathrm{id}$ $\quad \hat{\beta}\hat{\alpha} = \mathrm{id}$

So $\hat{\alpha}$ is ~~injec~~ bijective. $\square$

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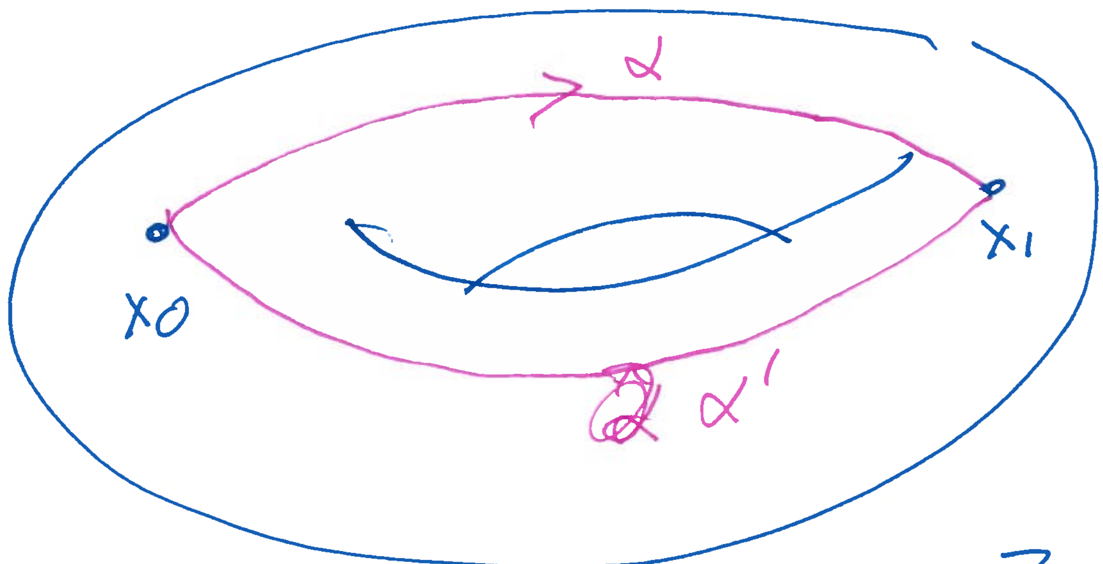
So if $X$ is <u>path connected</u> the isomorphism class of $\pi_1(X, x_0)$ is ~~well defined~~ independent of basepoint

<u>Warning</u>: The isomorphism

$$\pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

~~des~~ depends on the arc $\alpha$.



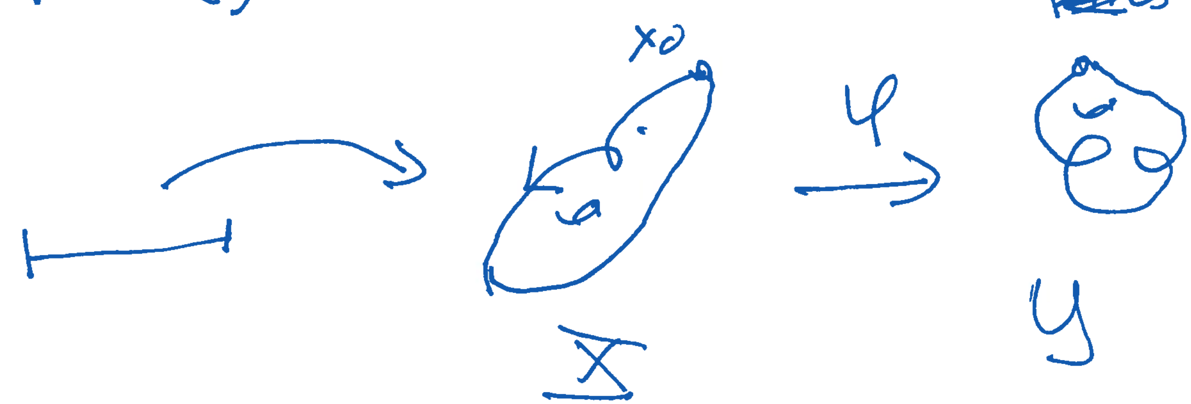

$$\pi_1(T^2, x_0) \cong \mathbb{Z}^2$$

$$\pi_1(T^2, x_1) \cong \mathbb{Z}^2$$

$\varphi : X \longrightarrow Y$ continuous, $f : I \to X$

Define $\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, \varphi(x_0))$

via $\varphi_*([f]) = [\varphi \circ f]$




is a homomorphism.