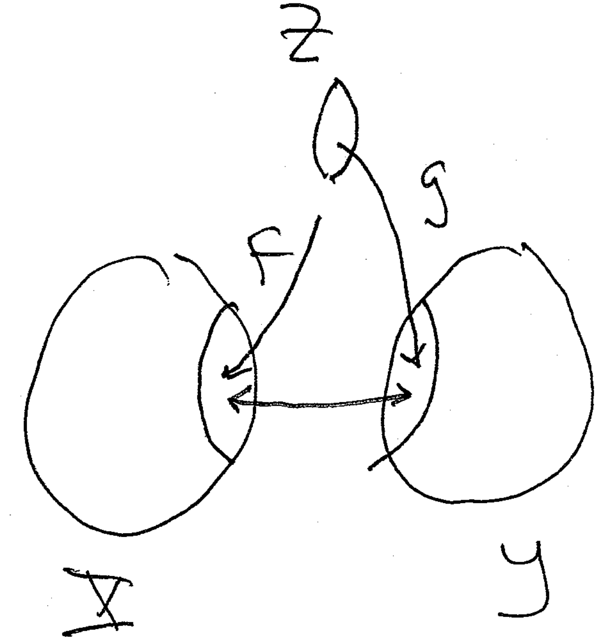
More on pushouts

Pushout in TOP.

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, dashed] \\ Y \arrow[r, dashed] & P \end{tikzcd}$$



$P = X \cup Y / \sim$

$f(z) \sim g(z)$ in $f(Z) \cup g(Z)$

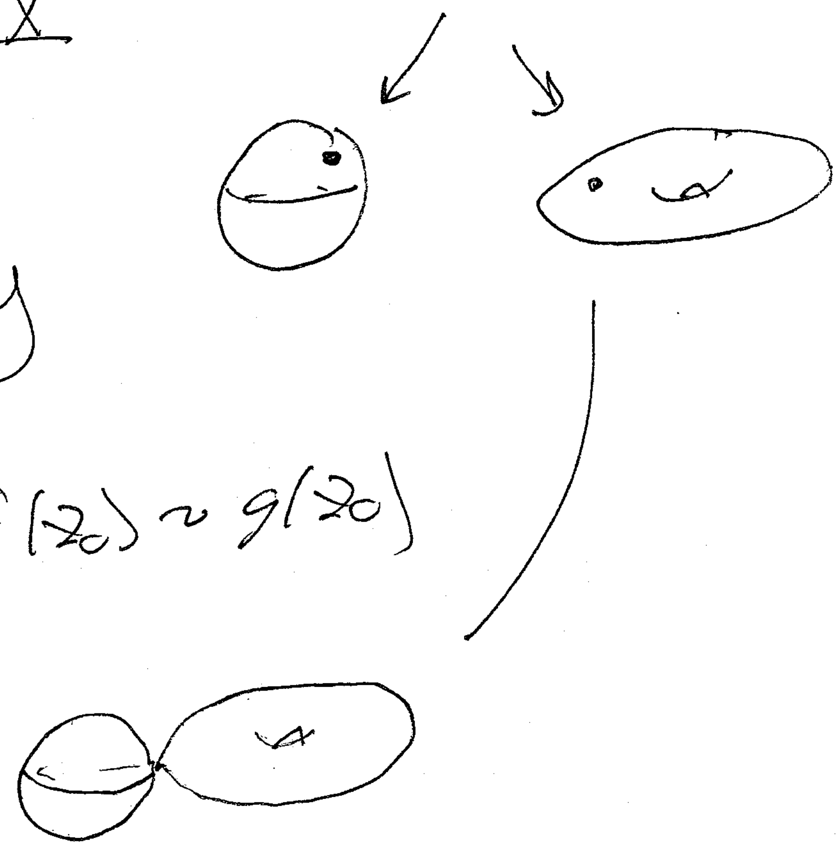
$x \sim x$ otherwise

with quotient topology.

Special cases

(1) $Z = \{z_0\}$ single pt

$$\begin{array}{ccc} & \xrightarrow{f} & X \\ \{z_0\} & & \\ & \xrightarrow{g} & Y \end{array}$$



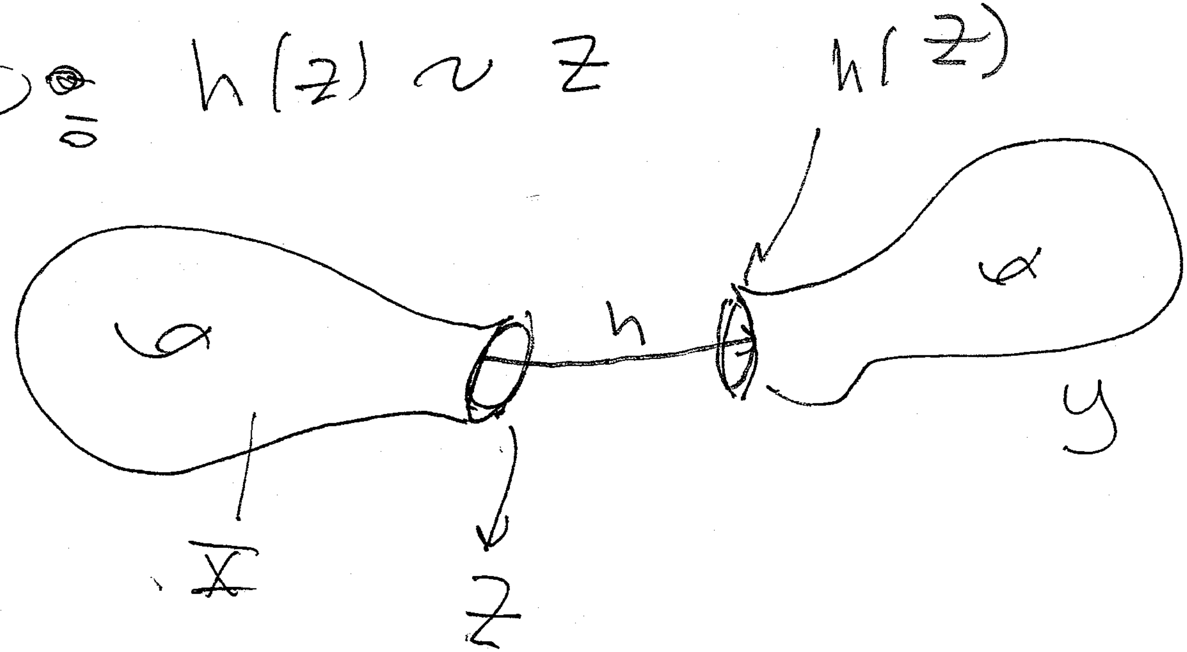
$X \vee Y / \sim \quad f(z_0) \sim g(z_0)$

$P = X \vee Y$

(2)

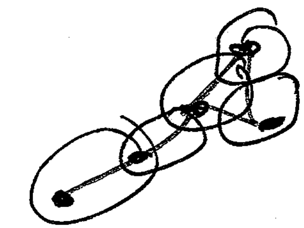
$$\begin{tikzcd} Z \arrow[r, phantom, "\subseteq"] \arrow[dr, "h"'] & X \arrow[d, dashed, bend left, "P"] \\ & Y \end{tikzcd}$$

$P \doteq h(Z) \sim Z$

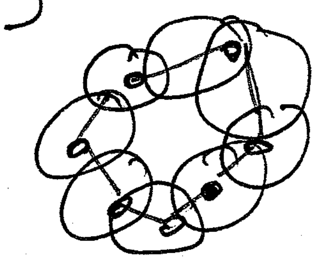


TDA con't — idea

DATA $\mathcal{S}$ $\{x_0, \ldots, x_n\} \subseteq \mathbb{R}^N$



Fatten up $X_\varepsilon = \bigcup_{i=0}^{n} B_\varepsilon(x_i)$

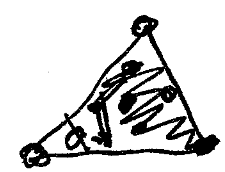


Approximate X_ε by a combinatorial object (simplecial complex) that is homotopic to X_ε and thus has the same algebraic topology.

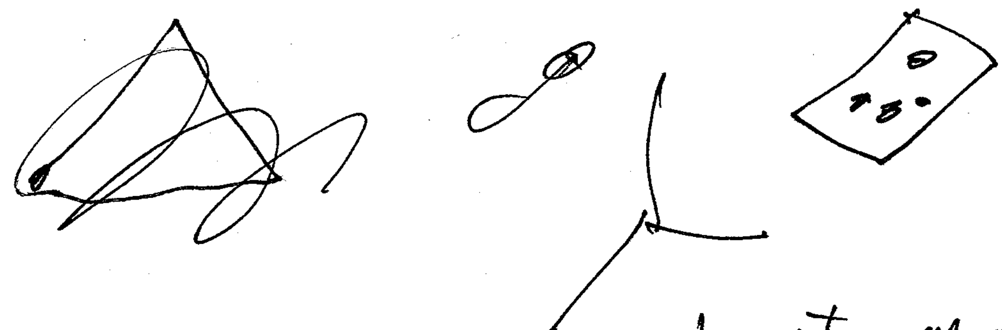
DEF of simplices and simplicial complex.

Given $\{x_0, \ldots, x_k\} \subseteq \mathbb{R}^N$ their simplicial span is

$$\Big\{ \underbrace{\sum_{i=0}^{k} a_i x_i}_{\text{convex combination}} : \sum a_i = 1 \quad a_i \geq 0 \Big\}$$



issue



4 pts in same plane don't make a 3-dim simplex.

__DEF__ $\{x_0, \dots, x_k\}$ are geometrically independent $\Leftrightarrow$ the vectors

$\{x_1 - x_0, x_2 - x_0, \dots, x_k - x_0\}$ are

linearly ind. in the Linear Algebra sense.

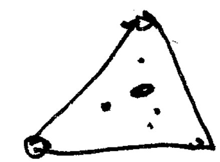
x_0

__RK__: geom ind $\Rightarrow$ $k \leq N$

Lemma $\{x_0, \ldots x_k\}$ is geom ind

$\Leftrightarrow$ $\forall x \in$ simplicial span

there $\exists!$ $a_i \geq 0$ with



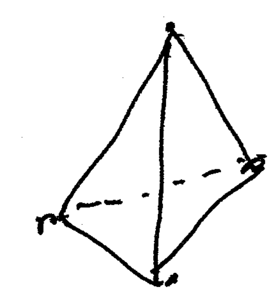
$$x = \sum a_i x_i$$

DEF: These are called the barycentric coordinates.

DEF Given $V = \{x_0, \ldots, x_k\}$ & geom. ind. $\Rightarrow$ the k-simplex they span is

$$S(V) = \left\{ \sum_{i=0}^{k} a_i x_i : \sum a_i = 1 \ a_i \geq 0 \right\}$$

$\dim(S(V)) = k$.



Remarks (a) $S(V)$ is the convex hull of V.

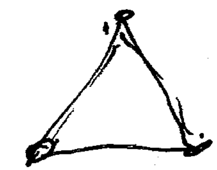
(b) $S(V)$ is cpt, connected.

in fact $S(V) \cong D^k$

$= \{ x \in \mathbb{R}^k : \|x\| \leq 1 \}$.

DEF: (a) $\mathrm{Int}(S(v))$ = topological interior

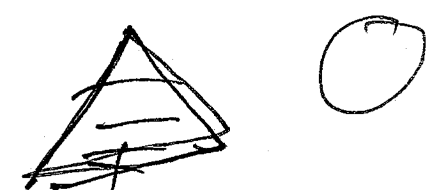
$= \text{all } \sum a_i x_i$ with all $a_i > 0$.



(b) $\mathrm{Bd}(S(v)) = S(v) - \mathrm{Int}(v)$

$= \text{all } \sum a_i x_i$ with at least one $a_i = 0$

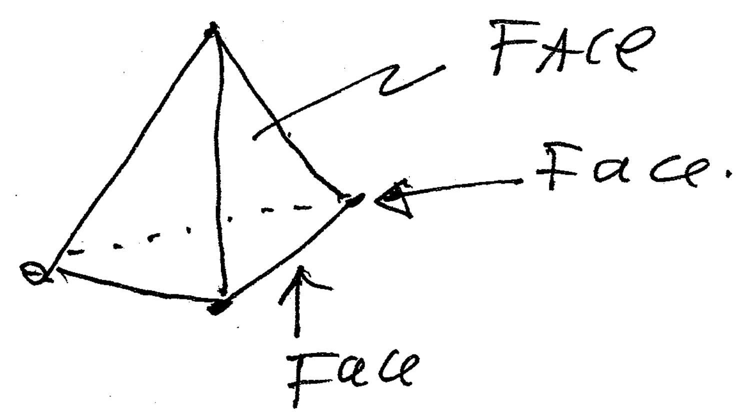
= Topological frontier of $S(v)$



RK $\mathrm{Bd}(S(v)) = S^{k-1}$

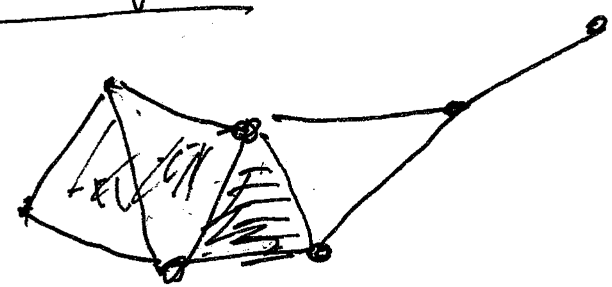
2-simplex

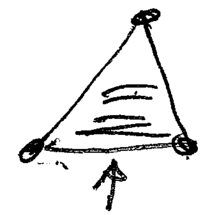
DEF A face is a simplex spanned by a proper subset of V (corresponds to some set of $a_i = 0$)



Lemma: $Bd(S)$ = union of its faces.

Simplicial complex

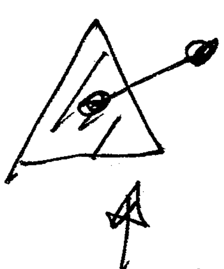




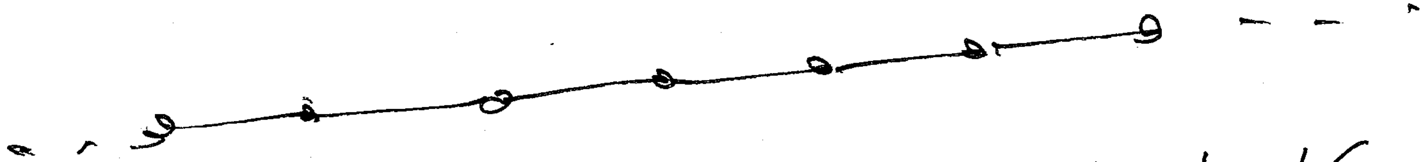
DEF: A simplicial complex ~~K~~ K is a set of simplices in $\mathbb{R}^N$ such that

(1) every face of a simplex in K is also a simplex in K

(2) The intersection of two simplicies is a face of one of them.



not this

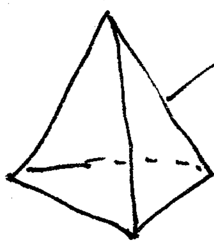


Standing Assumption: Is that K has only finitely many simplicies.

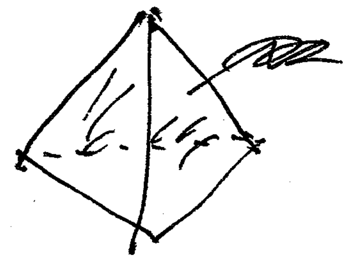
NOTE: **infinite case** also much studied.

DEF Given a simplicial complex K ~~its geo~~. Then $|K|$ is the subset of $\mathbb{R}^N$ given by the union of simplicies with the subspace topology.

DEF The p-skeleton of K, denoted $K^{(p)}$, is all simplices in K with dimension at most p. $K^{(0)}$ is the set of vertices.

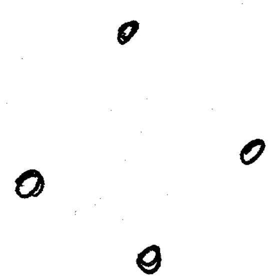


$K^{(1)}$

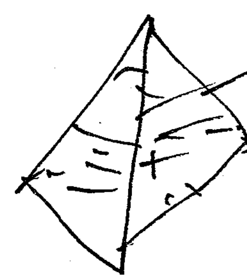


~~$K^{(3)} = K$.~~

~~$K^{(2)}$~~



$K^{(0)}$



$K^{(2)}$ Triangular faces

$K^{(3)} = K$.

The collection of simplicial complexes
are objects in a category
the arrows or morphisms are
simplicial maps — next time.