Recall:

- $V = \{x_0, \ldots, x_k\} \subseteq \mathbb{R}^N$ geometrically ind., the k-simplex spanned by V is

$$S(V) = \left\{ \sum_{i=0}^{k} a_i x_i : a_i \geq 0 \quad \sum a_i = 1 \right\}$$

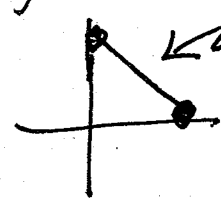
- FACE = span of a proper subset of V

- standard k-simplex

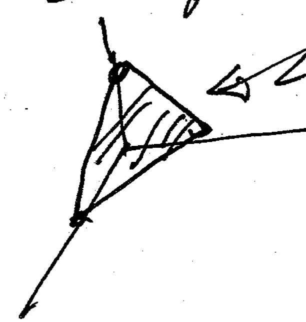
$$\Sigma_k = \{ \vec{e}_0, \ldots, \vec{e}_{k} \} \text{ with } \vec{e}_i$$

the std basis vectors in $\mathbb{R}^{k+1}$

$S(\Sigma_k)$ is the std k-simplex



$S(\Sigma_1)$

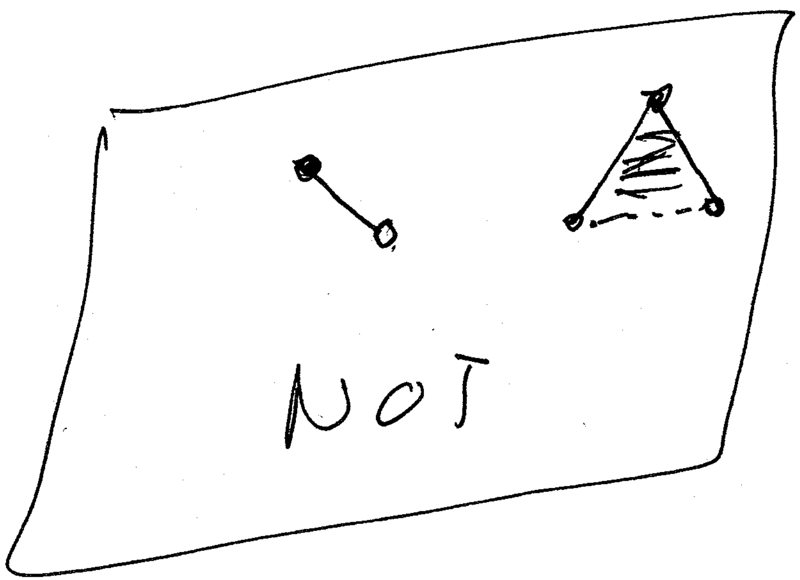


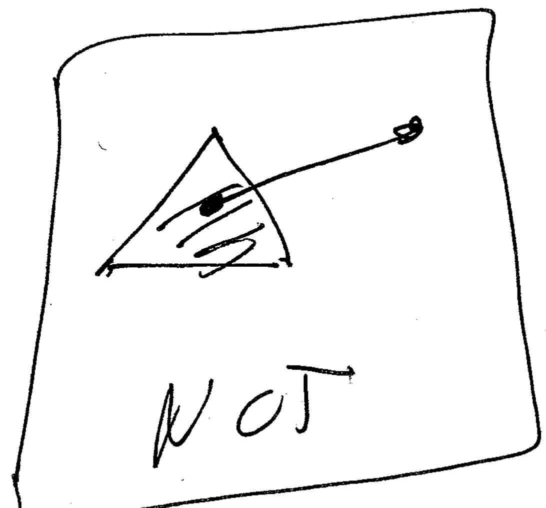
$S(\Sigma_2)$

- A finite simplicial complex K is a finite union of simplices in $\mathbb{R}^N$ for some N such that

(1) If σ is a simplex in K $\Rightarrow$ all its faces are simplices in K

(2) nonempty Intersection of two simplices is a face of each.

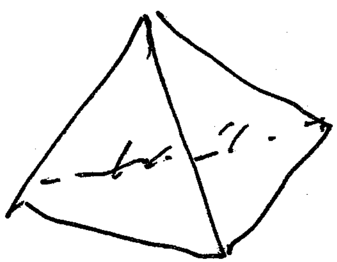




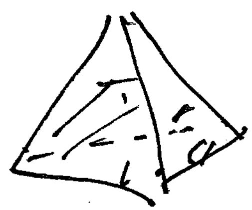
DEF: The geometric realization of K, denoted $|K|$, is the topological space in $\mathbb{R}^N$ which is the union of simplices with the subspace topology.

RK: $|K|$ is cpt.

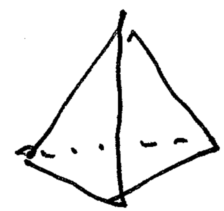
DEF: $K^{(p)}$ = union of all simplices of $\dim \leq p$ = "p-skeleton"



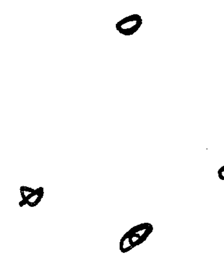
$K^{(3)}$



$K^{(2)}$

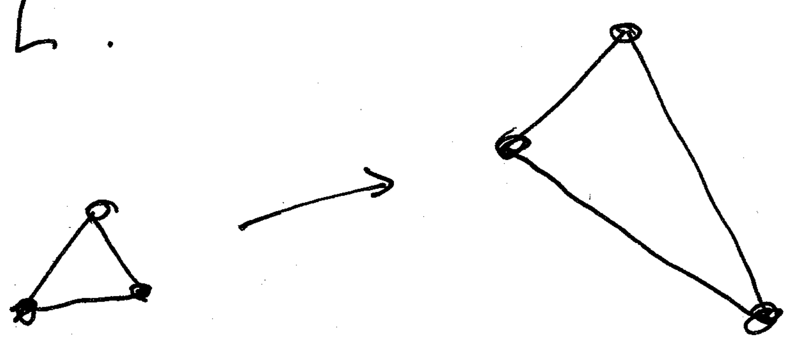


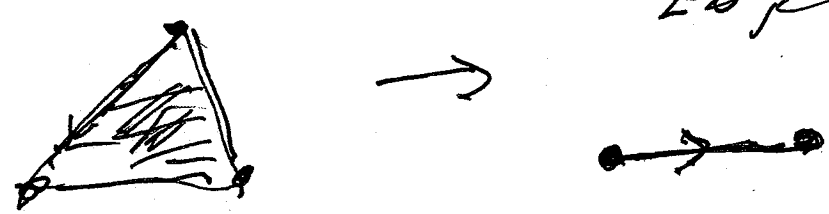
$K^{(1)}$



$K^{(0)}$

DEF K and L are simplicial complexes $f: K \to L$ is a simplicial map if whenever $\{x_0, \ldots, x_\ell\}$ spans a simplex in K $\Rightarrow$ $\{f(x_0), \ldots, f(x_\ell)\}$ spans a simplex in L.





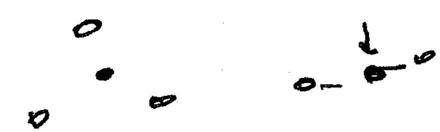
$\{x_0, x_1, x_2\} \to \{z_0, z_1\}$

$x_0 \to z_0$
$x_1 \to z_0$
$x_2 \to z_1$

<u>RK</u>: simplicial complexes with simplicical maps forms a category.

<u>Lemma</u> A simicial map $f: K \to L$ induces a continuous function $f: |K| \to |L|$ via the formula

$\sum a_i x_i \in \sigma$



define $f(\sum a_i x_i) = \sum a_i f(x_i)$

<u>Proof</u>: cont on simplices. (check non-injective case) – pasting lemma on intersections.

__FACT:__ geometric realization is a functor

$$\mathrm{Simp} \longrightarrow \mathrm{Top}.$$

(not __onto__)

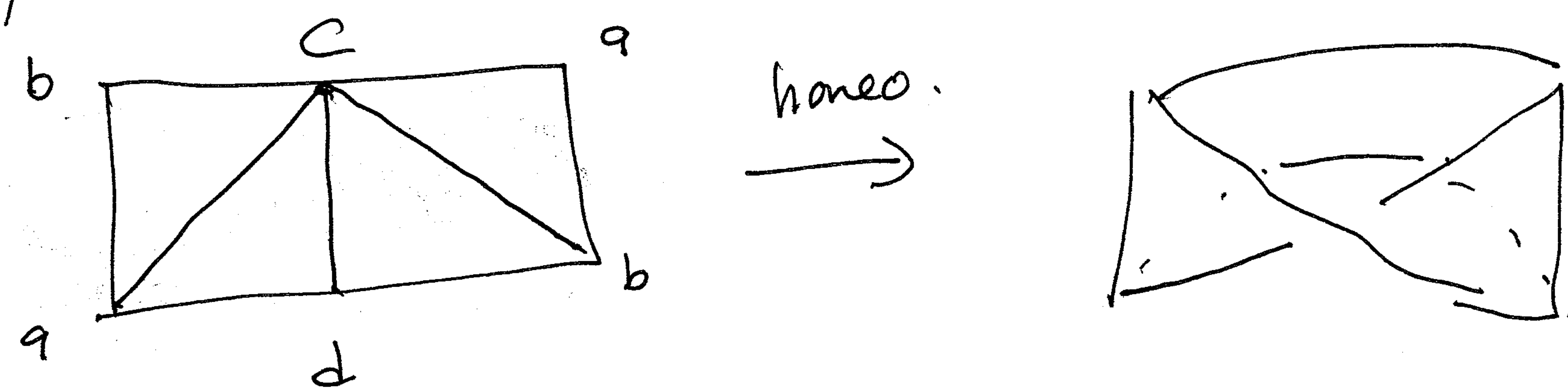
__RK:__

- A simplicial map is an isomorphism in Simp when $\{x_0, \dots, x_e\}$ spans a simplex in $K \Longleftrightarrow \{f(x_0), \dots, f(x_e)\}$ " " " in L.
- And if $f : K \to L$ is an isomorphism $\Rightarrow |f| : |K| \to |L|$ is a homeomorphism

__DEF:__ A top space $\underline{X}$ is called triangulatable if $\exists$ a finite simplicial complex K and a homeomorph ~~me~~ $h: K \to \underline{X}$.

__RK__ ~~AS~~ All ~~su~~ compact surfaces, 3-mfd and smooth mfds can be triangulated. But $\exists!$ a topological 4-mfd that is not.

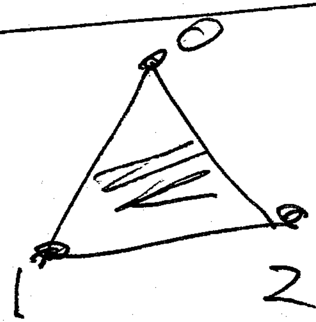
es//



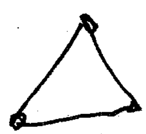
Notice – a simplicial complex is specified by its vertices and which vertices are involved in simplicies.

Leads to abstract simplicial complex.

DEF: Given $X = \{x_0, \ldots, x_k\}$ an abstract simplicial complex A with vertex set X is a collection of subsets of X with the property that every subset of a subset is also in the collection "closed under subsets" and $\{x_i\} \in A \ \forall i$.



$A = \{\{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{2,0\},$
$\{0,1,2\}\}$

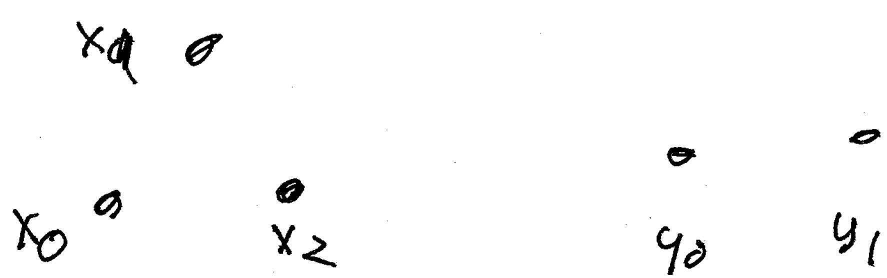


drop

DEF: An abstract simplicial map from A with vertex set $\{x_0, .., x_k\} = X$ and B with a vertex set $\{y_0, .., y_\ell\} = Y$ a map $f: X \longrightarrow Y$ such that

if $\{x_{a_1}, .., x_{a_\ell}\} \in A$

$\Rightarrow \{f(x_{a_1}), .., f(x_{a_\ell})\} \in B.$



$$f: \begin{aligned} x_0 &\mapsto y_0 \\ x_1 &\mapsto y_0 \\ x_2 &\mapsto y_1 \end{aligned}$$

__RK:__ This makes abstract simp comp a category. Isomorphism as defined

---

__Lemma:__ Each simplicial complex K generates an abstract simplicial complex A

via $\{x_{a_1}, \ldots, x_{a_\ell}\} \in A$

$\Leftrightarrow$ $\{x_{a_1}, \ldots, x_{a_\ell}\}$ span a simplex in K

__Lemma__: For evg abstract simplicial complex A $\exists$ a simp com K which realizes it.

__Proof__ Say A has vertex set $\{x_0, x_1, \ldots, x_k\}$. define $K \subseteq \mathbb{R}^{k+1}$ with vertex set $\{\vec{e}_0, \ldots, \vec{e}_k\}$ and simplex $\{\vec{e}_{a_1}, \ldots, \vec{e}_{a_\ell}\} \iff$

$\{\cancel{a}\, x_{a_1}, \ldots, x_{a_\ell}\} \in A$.

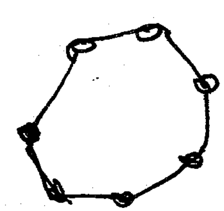
Abstract Simp $\longleftrightarrow$ Simplic complexe $\longrightarrow$ Top

$\longleftarrow$?

Given Top space ~~has~~ can be homeomorphic to many simplicial complexes.

$A = \{\{x_0\}, \ldots, \{x_n\}, \{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_n, x_0\}\}$

$\downarrow$



Circle.

How do you decide when two simp comp are homeomorphic (~~te~~ triangulate the same space) or even homotopic?

~~Let S be a simplic~~