Simplices $\xrightarrow{\text{functor}}$ Top spaces

Simplicial map $\qquad$ and cont. func ?

$$K \longrightarrow |K|$$

$$f: K \to L \longrightarrow |f|: |K| \to |L|$$

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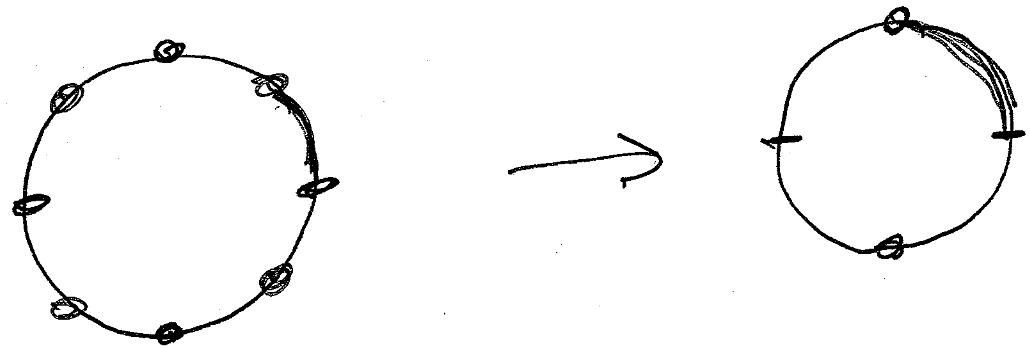
o ther way.

1. Many spaces are homeomorphic to a simplicial complex (triangulated)

2. What about cont funct?

Soln: Subdivide the complex - in the domain



$S^1 \qquad\qquad S^1$

With the new subdivided simpl com in the domain, f is simplicial.

Simplicial Approximation Theorem

K and L finite simp comp.

$|f|$ $f : |K| \to |L| \Rightarrow \forall \varepsilon > 0$

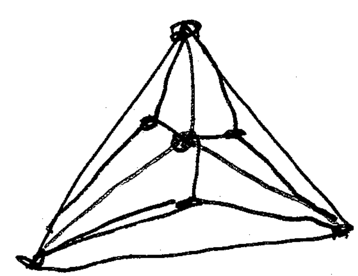
$\exists$ a subdivision K' of K

and a simplicial map $f' : K' \to L$

with (1) $|f'|$ homotopic to f

(2) $d(|f'|, f) < \varepsilon$.

Barycentric Subdivision



$$d(f,g) = \cancel{\inf}\ \min\{d(f(x), g(x)) : x \in K\}$$

Homology (Simplicial homology)
(Also singular homology, de Rham cohom)

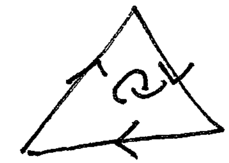
For each $n \geq 0$ and coeff
Ring R (Euclidian) $\Rightarrow$
H_n is a functor
$H_n : Simp \to Mod_R$ (modulues over R)
H_n counts the # of n-dim holes in X
Most common. $R = \mathbb{R} \Rightarrow Mod_R = Vect(\mathbb{R})$
$R = \mathbb{Z} \Rightarrow Mod_R = Ab$

1st step order simplices —

Informally

$[x_0, x_1]$

$[x_1, x_0]$

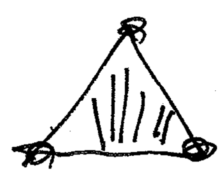


<u>DEF</u> ~~A~~ an ordering of the simplex spanned by $\{x_0, .., x_k\}$ is an equiv class of orderings of vertices with two equiv if they differ by an even permutation.

$[x_0, .., x_k]$ denotes an ordered simplex.

DEF Fix a simplicial X and ring R the k-chains $C_k(X;R)$ is the free R-module on the set of k-simplicies

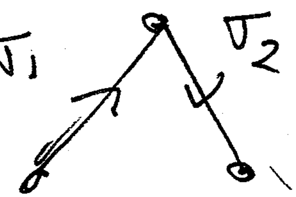
$$C_k(X;R) = \underbrace{R \oplus \cdots \oplus R}_{\#\text{ of k-simplicies}}$$



$$C_0 = R^3 \qquad \mathbb{R}^3 \text{ or } \mathbb{Z}^3$$
$$C_1 = R^3 \Rightarrow$$
$$C_2 = R$$

convention: σ is an ordered simplex
σ' is the same with opp orientation.

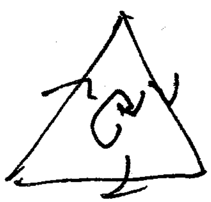
In C_K, we define $\cancel{\sigma'}$ $-\sigma' = \sigma$



$\sigma_1 + \sigma_2$ $\Rightarrow$ chain

$3\sigma_1 + 2\sigma_2$

$\leftarrow$ Bd isn't top visible since it is the Bd of a simplex in $\underline{X}$.



not the Bd, so contribute to H_1

DEF: The $\partial_k : C_k(X;R) \to C_{k-1}(X;R)$

$$\leftarrow C_{k-1} \xleftarrow{\partial_k} C_k \leftarrow$$

defined on generators (i.e. oriented simplices)

by

$$\partial_k([x_0 \ldots x_k]) = \sum_{i=0}^{k} (-1)^i [x_0, \ldots, \hat{x}_i, \ldots x_k]$$

eliminate i^{th} entry.

$\Rightarrow$ extend via linearity

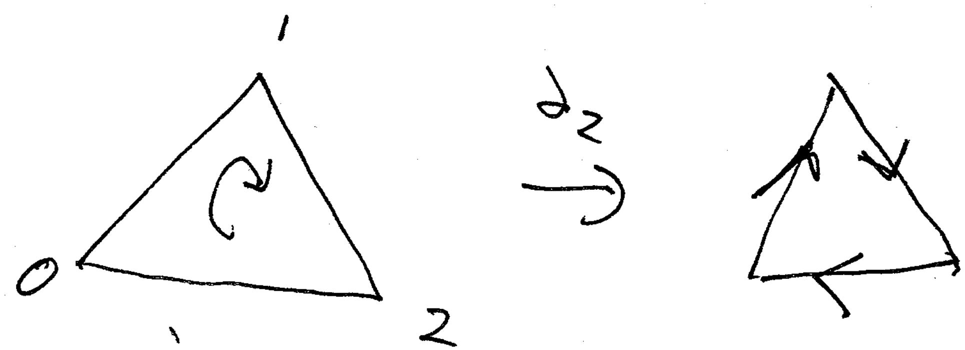
eg: $R = \mathbb{R}$, $\partial_k : \mathbb{R}^{\ell_k} \to \mathbb{R}^{\ell_{k-1}}$

Linear transformation.

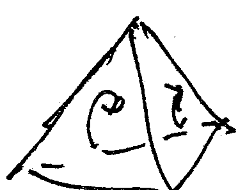
Rk: vertices have just one orientation

$$\partial_1 [v_0 v_1] = [v_1] - [v_0]$$

$$\partial_2 [v_0 v_1 v_2] = [v_1 v_2] - [v_0 v_2] + [v_0 v_1]$$



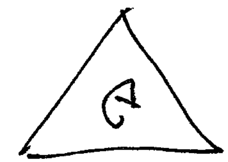
$$\partial_3 [v_0 v_1 v_2 v_3] = [v_1 v_2 v_3] - [v_0 v_2 v_3] + [v_0 v_1 v_3] - [v_0 v_1 v_2]$$



$$C_0 \leftarrow \cdots \leftarrow C_{k-1} \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1} \leftarrow \cdots$$

<u>**Basic Fact**</u>: $\partial_k \circ \partial_{k+1} = 0$

(Bd of Bd is empty - algebraically)

$$\partial_2([v_0 v_1 v_2]) = [v_1 v_2] - [v_0 v_2] + [v_0 v_1]$$

$$= v_2 - v_1 - (v_2 - v_0) + (v_1 - v_0)$$

$\partial_{\cancel{3}} \partial_2$

$$= 0$$

DEF: In general, a collection of R-modules with connecting homomorph ∂_k

$$C_0 \leftarrow \cdots \leftarrow C_{k-1} \xleftarrow{\partial_k} C_k \leftarrow$$

is called a chain-complex if

$$\partial_k \circ \partial_{k+1} = 0$$

NOTICE: This happens exactly when

$$\operatorname{Im} \partial_{k+1} \subseteq \ker \partial_k$$

<u>DEF</u>: $\operatorname{Im} \partial_{k+1} = k$- ~~∂~~ boundaries $= B_k$

$\ker \partial_k = Z_k = k$-cycles (Zyklus)

The ~~⊕~~ k^{th} ~~co~~homology of the chain complex $\mathcal{C}$ is

$$H_k(\mathcal{C}) = \ker(\partial_k) \big/ \operatorname{Im}(\partial_{k+1}) = Z_k / B_k$$

~~Special~~ When $\mathcal{C}$ comes from a simplicial complex X, this is written

$$H_k(X; R)$$

Two special cases

$H_k(X;\mathbb{R})$ is a finite dimentional vector space. since $\mathbb{R}^k/\mathbb{R}^\ell \cong \mathbb{R}^{k-\ell}$

$H_k(X;\mathbb{Z})$ is a finitely generated Abelian group. or

$$\underbrace{\mathbb{Z}^{\beta_k}}_{\text{Free}} \oplus \underbrace{\left(\oplus \mathbb{Z}_{n_i}\right)}_{\text{Torsion}}$$

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e.g. // $Z_k = \mathbb{Z}$ $\quad H_k = \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3$

$B_k = 3\mathbb{Z}$