DEF: A chain complex of R-modules is a collection $\{C_i\}_{i=-\infty}^{\infty}$ of R-modules with connecting ~~maps~~ homomorphisms

$$\partial_k : C_k \to C_{k-1}$$ such that

$$\partial_{k+1} \circ \partial_k = 0$$

$$\boxed{\leftarrow C_{-1} \leftarrow C_0 \leftarrow C_1 \leftarrow \cdots}.$$

or

$$\begin{array}{c} \downarrow \\ C_1 \\ \downarrow \partial_1 \\ C_0 \\ \downarrow \partial_0 \\ C_{-1} \\ \downarrow \end{array}$$

For simplicial homolog.

$\underline{C_k = C_k(X; R)}$ which is

the k-chains of the simplicial complex X

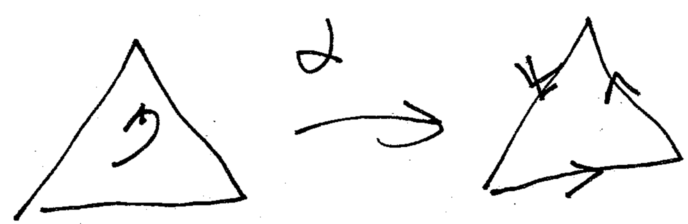
i.e. free R-module on oriented simplices.

$C_k = 0$ $k < 0$, $C_k = 0$ $k > \dim(X)$

= ~~dimension~~ top dimension of any simplex in X.

$$\partial_k [v_0 \ldots v_k] = \sum_{i=0}^{k} (-1)^i [v_0 \ldots \hat{v}_i \ldots v_k]$$

eliminate



Since $\partial_{k+1} \circ \partial_k = 0 \Rightarrow \operatorname{im} \partial_{k+1} \subseteq \ker \partial_k$

the k^{th} homology is

$$\ker \partial_k / \operatorname{im} \partial_{k+1} = H_k(\mathcal{C}) = H_k(X; R)$$

general chain complex

$\ker \partial_k = Z_k = k$- ~~cycles~~ cycles

$\operatorname{im} \partial_{k+1} = B_k = k$- boundaries

$R = \mathbb{R}$, $H_k = R^{\beta_k}$ vectorspace

$R = \mathbb{Z}$, $H_k = (\bigoplus \mathbb{Z}) \oplus \underbrace{(\bigoplus \mathbb{Z}_{a_i})}_{\text{torsion}}$

Morphisms in the cat of chain complexes.

DEF: Given two chain complexes $\mathcal{C}$ and $\mathcal{D}$ both over R a family $\varphi_n : C_n \to D_n$ is ~~ch~~ called a chain map

$$\begin{tikzcd} \arrow[d] & \arrow[d] \\ C_n \arrow[r, "\varphi_n"] \arrow[d, "\partial_n^A"'] & D_n \arrow[d, "\partial_n^D"] \\ C_{n-1} \arrow[r, "\varphi_{n-1}"] \arrow[d] & D_{n-1} \arrow[d] \\ {} & {} \end{tikzcd}$$

commutes

or $\partial_n^B \circ \varphi_n = \varphi_{n-1} \circ \partial_n^A$.

__Remark__: R-module chain complexes with chain maps is a category.

__Lemma__: A simplicial map $f: K \to L$ induces a chain map

$$C_*(K,R) \to C_*(L,R)$$

the chain complex for K

__Proof__: $\left.\begin{array}{l} f_n([v_0 \ldots v_e]) \\ = \underbrace{[f(v_0) \ldots f(v_e)]}_{\text{cancelling repitions}} \end{array}\right\}$ on basis elements

<u>Hw:</u> A chain map $\varphi: \mathcal{C} \to \mathcal{D}$

induces a homomorphism

$$\varphi_* : H_*(\mathcal{C}) \to H_*(\mathcal{D})$$

$$\left[\text{i.e. } \forall n, \quad \varphi_n : H_n(\mathcal{C}) \to H_n(\mathcal{D}) \right]$$

<u>Functors</u>

Simplices $\longrightarrow$ chain complexes $\longrightarrow$ Homology.

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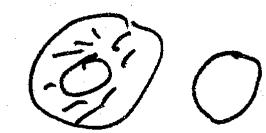
homotopy? If K is ~~homotopic~~ homotopy equiv to L 

is $H_*(K) \cong H_*(L)$.

DEF: $f, g: \mathcal{C} \to \mathcal{D}$ two chain maps are <u>chain homotopic</u> if

$\exists\, h_n: C_n \to D_{n+1}$

with $f_n - g_n = \partial_{n+1}^D \circ h_n - h_{n-1} \circ \partial_n^C$

$$\begin{tikzcd}
C_{n+1} \arrow[d] & D_{n+1} \arrow[d, "\partial_{n+1}^D"] \\
C_n \arrow[d, "\partial_n^C"'] \arrow[ur, "h_n"] & D_n \arrow[d] \\
C_{n-1} \arrow[ur, "h_{n-1}"] & D_{n-1}
\end{tikzcd}$$

HW: chain homotopic maps induce the same maps on Homology i.e.

$$f_* = g_* \text{ as } H_*(\mathcal{C}) \to H_*(\mathcal{D}).$$

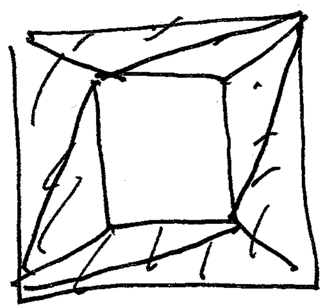
FACT: $f, g : K \to L$ simplicial maps with $|f|, |g| : |K| \to |L|$ are homotopic

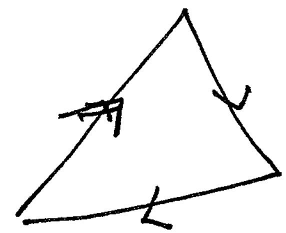
$\Rightarrow$ their maps on $\mathcal{C}_*(K) \to \mathcal{C}_*(L)$ are chain homotopic

$\Rightarrow$ $f_*, g_* :$ $(\mathcal{C})$
$H_*(\mathcal{C}) \to H_*(\mathcal{D})$
are equal.

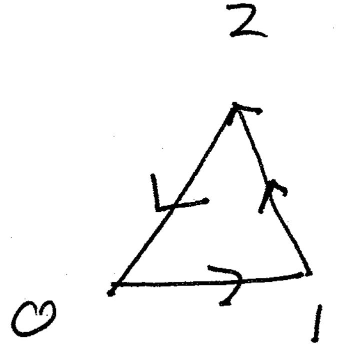
Conclusion: If K is homotopy equiv to L

$\Rightarrow H_*(K) = H_*(L)$.





Same H_*.



$\{0, 1, 2, 01, 12, 20\}$

$$C_{-1} \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \leftarrow$$

$$0 \xleftarrow{\partial_0} \mathbb{R}^3 \xleftarrow{\partial_1} \mathbb{R}^3 \xleftarrow{\partial_2} 0$$

$$H_0 = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \mathbb{R}^3 / \operatorname{im} \partial_1$$

$$H_1 = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\ker \partial_1}{0} = \ker(\partial_1)$$

Basis C_0: v_0, v_1, v_2

C_1 : $[v_0 v_1], [v_1 v_2], [v_2 v_0]$

$$\partial_1 [v_0 v_1] = v_1 - v_0$$

$$\partial_1 [v_1 v_2] = v_2 - v_1$$

$$\partial_1 [v_2 v_0] = v_0 - v_2$$

on $\partial_1 \left(a_0 [v_0 v_1] + a_1 [v_1 v_2] + a_2 [v_2 v_0] \right)$

$$= a_0 (v_1 - v_0) + a_1 (v_2 - v_1) + a_2 (v_0 - v_2)$$

$$= (a_2 - a_0) v_0 + (a_0 - a_1) v_1 + (a_1 - a_2) v_2 .$$

As a matrix

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 - a_0 \\ a_0 - a_1 \\ a_1 - a_2 \end{bmatrix}$$

$$\ker \partial_1 = \{ (a_0, a_1, a_2) : a_2 - a_0 = 0 = a_0 - a_1 = a_1 - a_2 \}$$

$$= \{ (a_0, a_1, a_2) : a_0 = a_1 = a_2 \}.$$

$$\dim(\ker \partial_1) = 1.$$

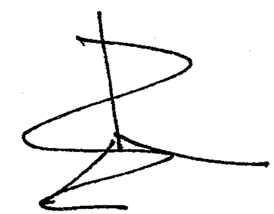
$$H_1 = \ker \partial_1 = \mathbb{R}$$

$\mathrm{Im}(\partial_{\ast}) = \mathrm{span}(a_2 - a_0, a_0 - a_1, a_1 - a_2)$

$\text{``}=\text{''}\ \mathrm{span}(\alpha, \beta, -\alpha-\beta)$

since sum $= 0$

$\cancel{\sum v_i = 0}$



$\dim \mathrm{Im}\, \partial_2 = 2.$

$$H_0 = \frac{\ker \partial_0}{\mathrm{Im}\, \partial_1} = \mathbb{R}^3 / \mathbb{R}^2 = \mathbb{R}^1.$$

$H_0 = \mathbb{R} \leftarrow$ one path component.

$H_1 = \mathbb{R} \leftarrow$ one one-dim hole