Computing simplicial homology

$$H_k = \ker \partial_k / \operatorname{Im} \partial_{k-1}$$

∂_k maps of free modules
so computing H_k is a linear alg problem.

TOOL IS SMITH NORMAL FORM
which works over any PID
eg $\mathbb{R}$, $\mathbb{Z}$. ($\mathbb{R}[t]$)

Th'm M is $m \times n$ matrix $\quad m\left[\overset{n}{}\right]$

$M: \mathbb{Z}^n \to \mathbb{Z}^m$

$\Rightarrow \exists$ SNF D of M with

$$D = \begin{bmatrix} d_1 & & & & 0 \\ & \ddots & & & \\ & & d_\ell & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix} \quad \boxed{\begin{matrix} m \times n \\ d_i > 0 \\ d_i \mid d_{i+1} \end{matrix}}$$

and invertible matrices A and B (over $\mathbb{Z}$)

with $M = ADB$ $\quad$ A is $\quad$ B is $n \times n$

A is $m \times m$.

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "B"] \arrow[rrr, bend right, "M"'] & \mathbb{Z}^n \arrow[r, "D"'] & \mathbb{Z}^m \arrow[r, "A"'] & \mathbb{Z}^m \end{tikzcd}$$

4 $\quad \mathbb{Z}^m/\mathrm{im}(D) = \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_\ell\mathbb{Z} \oplus \mathbb{Z}^{n-\ell}.$

$$= \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_\ell} \oplus \mathbb{Z}^{n-\ell}$$

where $\mathbb{Z}_1 = \mathbb{Z}/\mathbb{Z} = \vec{0}$

$\Rightarrow \mathbb{Z}^m/\mathrm{im}(M) =$ same.

(5) A is $m \times n$ $\quad$ B is $k \times m$

and $BA = 0$

$\Rightarrow$ Given SNF of A and B

$$\ker B / \operatorname{Im} A = \mathbb{Z}^{m-l_B} / \operatorname{Im} A$$

$$= \mathbb{Z}_{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{\alpha_{l_A}} \oplus \mathbb{Z}^{m-l_B-l_A}$$

α_i the diag on SNF(A).

Thm

$$C_{k-1} \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1}$$

$\partial_k \circ \partial_{k+1} = 0$ and $m_k = \dim(C_k)$

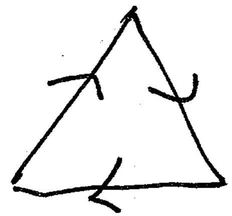
D_k is SNF of ∂_k.

$$\Rightarrow H_k = \mathbb{Z}_{d_{(k+1,1)}} \oplus \cdots \oplus \mathbb{Z}_{d_{(k+1,\, l_{k+1})}} \oplus \mathbb{Z}^{m_k - l(k) - l(k+1)}$$

CONCLUSION: Given the SNF of ∂_k

$\Rightarrow H_k$ is computed.

eg//



$$0 \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} 0$$

$$\cancel{0 \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \leftarrow 0}$$

$$0 \leftarrow \mathbb{Z}^3 \xleftarrow{\partial_1} \mathbb{Z}^3 \leftarrow 0$$

$$\partial_1 \longleftrightarrow \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{SNF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$m_2 = 0 \quad m_0 = 3 \quad m_1 = 3$

$\ell(1) = 2 \quad \ell(0) = \text{rank}(d_0) = 0$

$\ell(2) = 0$

$$H_0 = \mathbb{Z}^{m_0 - \ell(0) - \ell(1)} = \mathbb{Z}^{3-0-2} = \mathbb{Z}$$

$$H_1 = \mathbb{Z}^{m_1 - \ell(1) - \ell(2)} = \mathbb{Z}^{3-2-0} = \mathbb{Z}.$$

no torsion since all $d_i = 1$.

- For HW you can use computational platform for SNF

~~CAU~~ <u>WARNING</u>! SNF is slow and expensive - poly in dimension so efficient computation of homology is an active research area.

$$
\begin{array}{ccccc}
\text{Now DATA} & \longrightarrow & \text{Simplicial complex} & \longrightarrow & \text{Homology} \\
\downarrow & & \downarrow & & \downarrow \\
\text{at scale } \varepsilon & & \text{at scale } \varepsilon & & \text{for scale } \varepsilon
\end{array}
$$

$\boxed{\text{Persistent Homology.}}$

There are two basics methods to go from a finite data set to a simplicial complex

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$X = \{x_0, .., x_k\} \subseteq \mathbb{R}^N$ (or metric space)

distinct points

$B_\varepsilon(x_i) = \{y \in \mathbb{R}^N : d(y, x_i) < \varepsilon\}$

$y \subseteq X$

$B_\varepsilon(y) = y_\varepsilon = \{g : d(g, y) < \varepsilon\}$

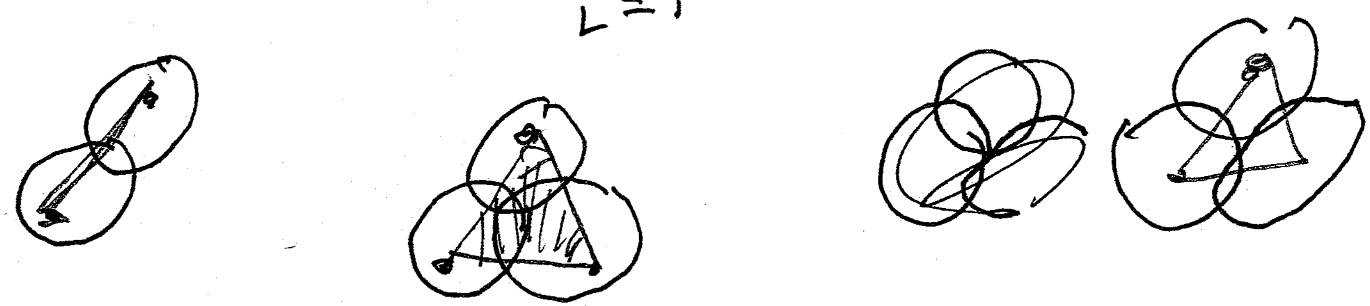
Given $\mathbb{X}$ and $\varepsilon > 0$ the Čech complex $\check{C}_\varepsilon(\mathbb{X})$ is the abstract simplicial complex with simplices

(1) vertices are the points of $\mathbb{X}$

(2) We have a simplex

$\{x_{a_1}, \ldots, x_{a_\ell}\}$

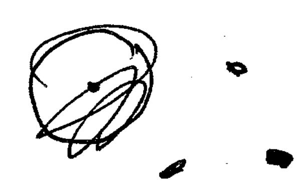
$$\Longleftrightarrow \bigcap_{i=1}^{\ell} B_\varepsilon(x_{a_i}) \neq \emptyset$$



Computationally ~~exp~~ expensive to ~~desc~~ decide when balls intersect.

The ~~altern~~ alternative more computable complex is Vietoris-Rips complex (VR)

DEF Given X and $\varepsilon > 0$. $VR_\varepsilon(X)$ has simplices



(a) vertices points of X

(b) $\{x_{a_1}, .., x_{a_\ell}\} \iff$

$$d(x_{a_i}, x_{a_j}) < 2\varepsilon \quad \forall\, i, j$$

(diameter of cluster of points)

Lemma $\check{C}_{\varepsilon}(X) \subseteq VR_{\varepsilon}(X) \subseteq \check{C}_{2\varepsilon}(X)$