Covering Spaces

$$\begin{array}{c} E \\ \downarrow p \\ B \end{array}$$

Top 2.5

is a covering space if

(1) p is continuous, onto

(2) $\forall x \in B$, $\exists$ nhd U of x

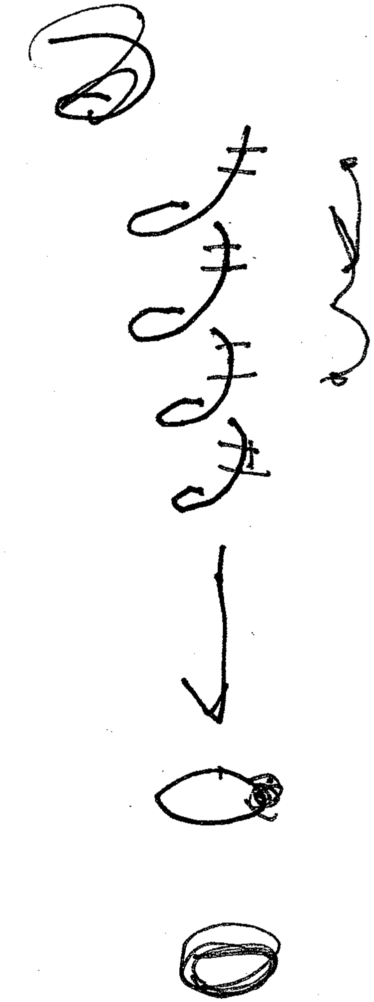
which is evenly covered i.e.

$$p^{-1}(y) = \coprod V_\alpha, \quad p|_{V_\alpha} \text{ is a}$$

homomeomorphism $\forall \alpha$.

Example) $p : \mathbb{R} \to S^1 \subseteq \mathbb{C}$

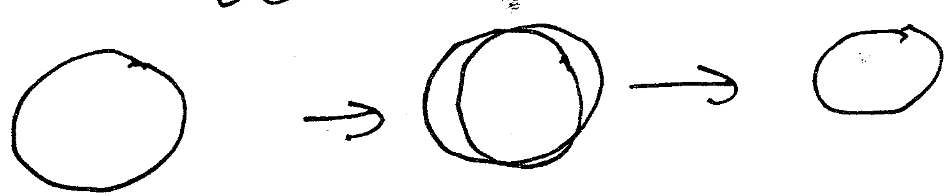
$$p(x) = e^{2\pi i x}$$

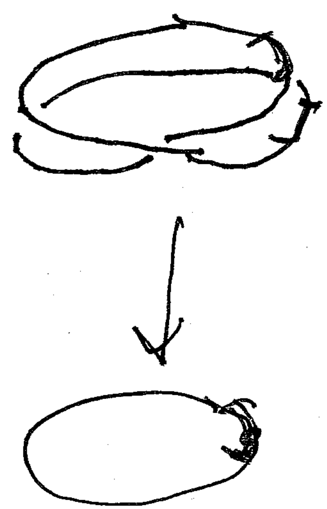


$p : S^1 \to S^1$

$p(z) = z^2$ (2-fold cover)

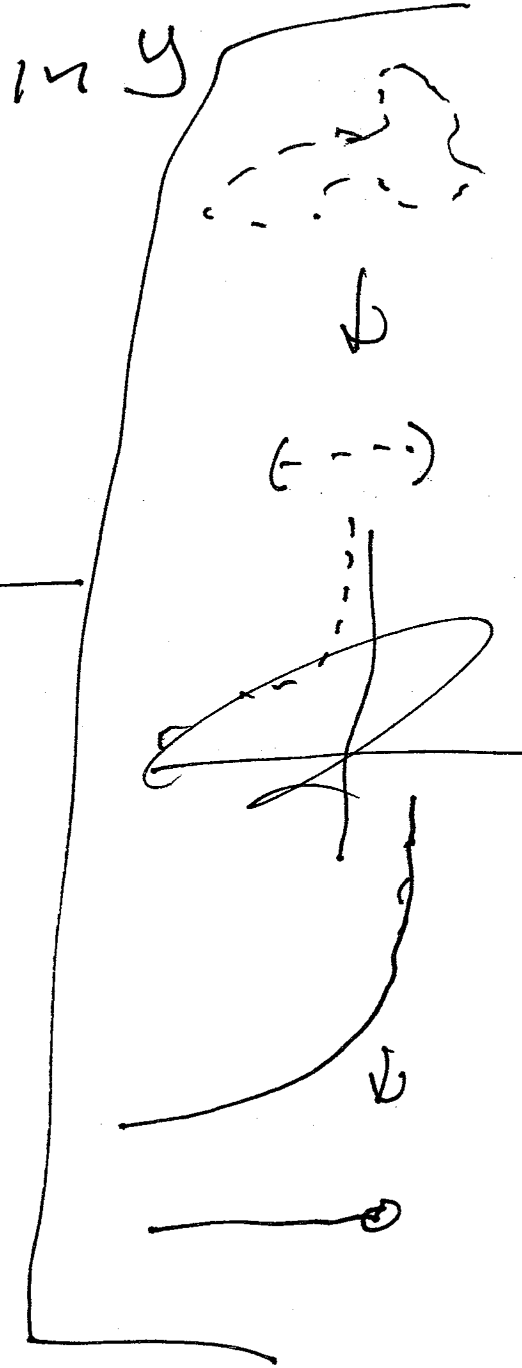
$(\theta \mapsto 2\theta)$





DEF: $f: X \to Y$ is an **open map** if $f(U)$ is open in Y $\forall$ open $U \subseteq X$

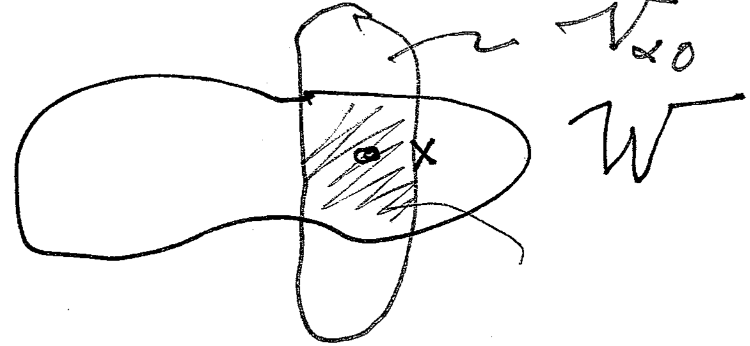
similar for **closed maps**

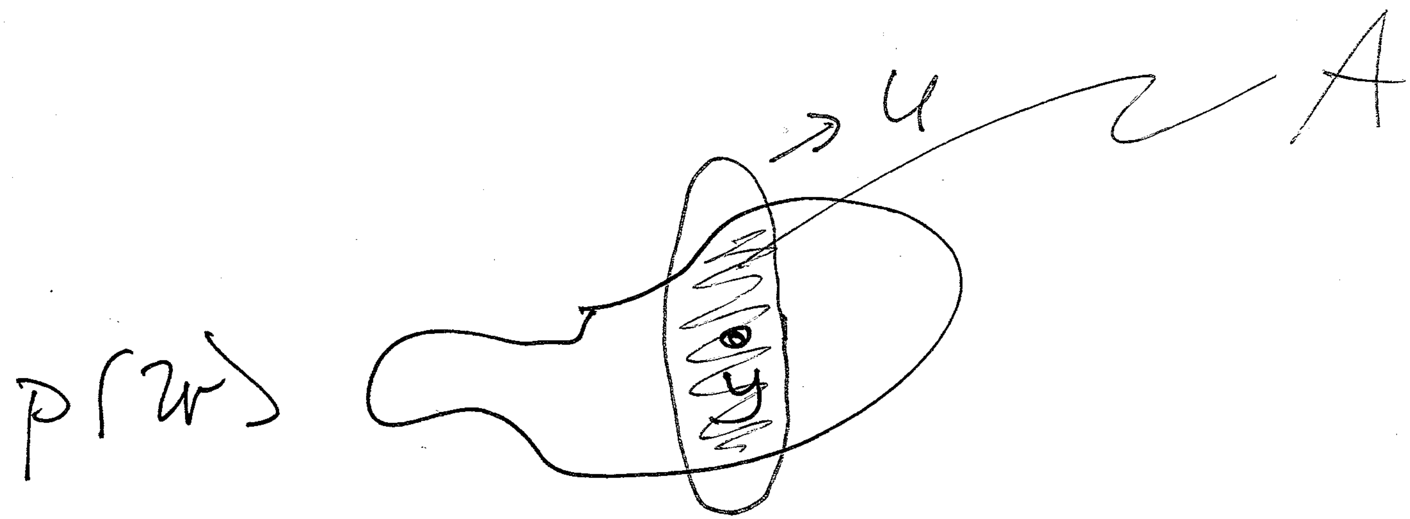


Lemma: $p: E \to B$ is a covering space then p is an open map.

Proof: Let $W \subseteq E$ open set we prove $p(W)$ is open by show that $\forall y \in p(W)$ $\exists$ open A with $y \in A \subseteq p(W)$

To prove this, $\exists$ nhd U of y that is evenly covers by $\{V_\alpha\}$. $\exists x \in W$ with $p(x) = y$ then $\exists \alpha_0$ with $x \in V_{\alpha_0}$ so $x \in V_{\alpha_0} \cap W$ which is open so $y \in p(V_{\alpha_0} \cap W)$ but $p|V_{\alpha_0}$ is a homeo an open set in $p(W)$. $\square$


V_{x_0}
W


U
A
y
$p(W)$

Thm: $p: E \to B$ is a covering space $B_0 \subseteq B$ let $E_0 = p^{-1}(B_0)$ then

$$E_0 \xrightarrow{p|E_0} B_0$$ is covering space

Proof: easy.

Thm: $p: E \to B$, $p': E' \to B'$ are C.S.

$\Rightarrow p \times p': E \times E' \longrightarrow B \times B'$ is a C.S.

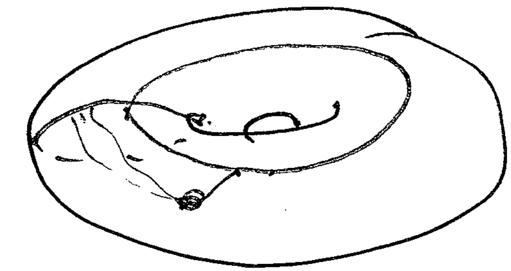
<u>Proof</u> ~~a~~ each $(b, b') \in B \times B'$ has U of b and U' of b' ~~with~~ ~~p~~ evenly covered by $\{V_\alpha\}$, $\{V'_\beta\}$

$$\Rightarrow (p \times p')^{-1}(U \times U') = \coprod_{(\alpha,\beta)} V_\alpha \times V'_\beta$$

and $(p \times p')\big|_{V_\alpha \times V_\beta}$ is a homeo $\forall (\alpha, \beta)$. $\blacksquare$

DEF $S^1 \times S^1$ is called $\mathbb{T}^2$, the 2-torus

$S^1 \times S^1 \subseteq \mathbb{C}^2 = \mathbb{R}^4$ but $\mathbb{T}^2$

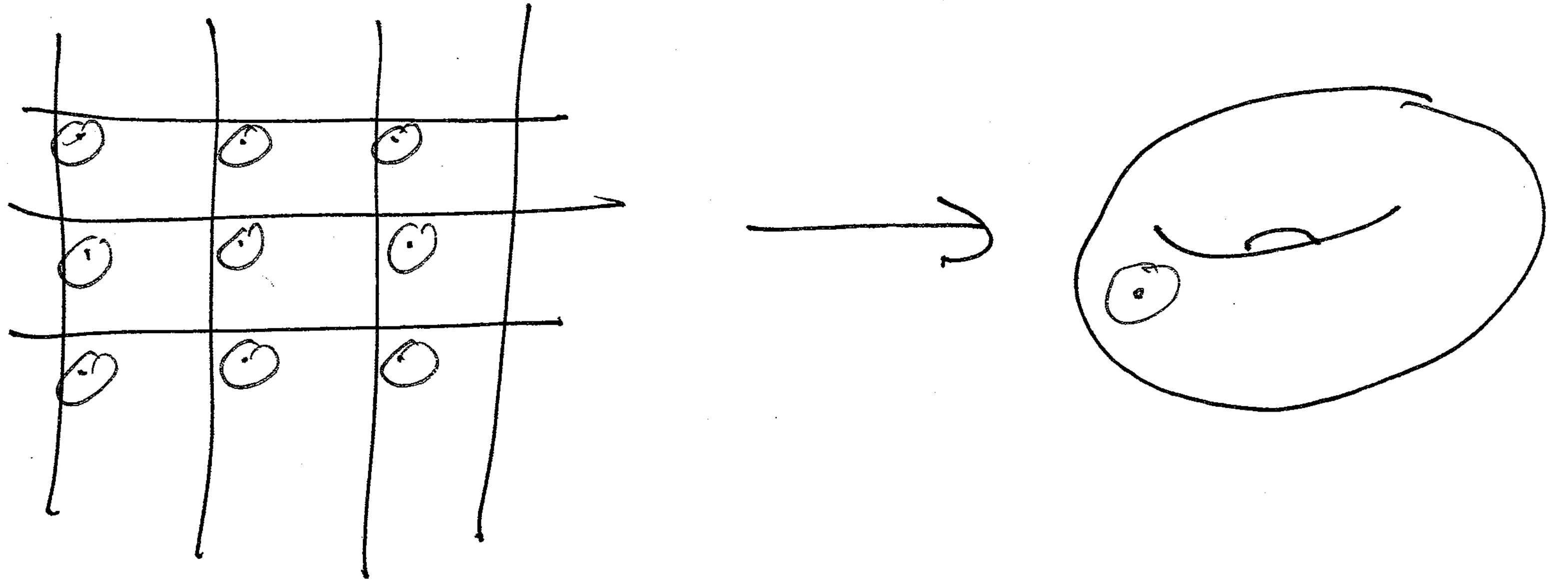
is homeomorphic to 

---

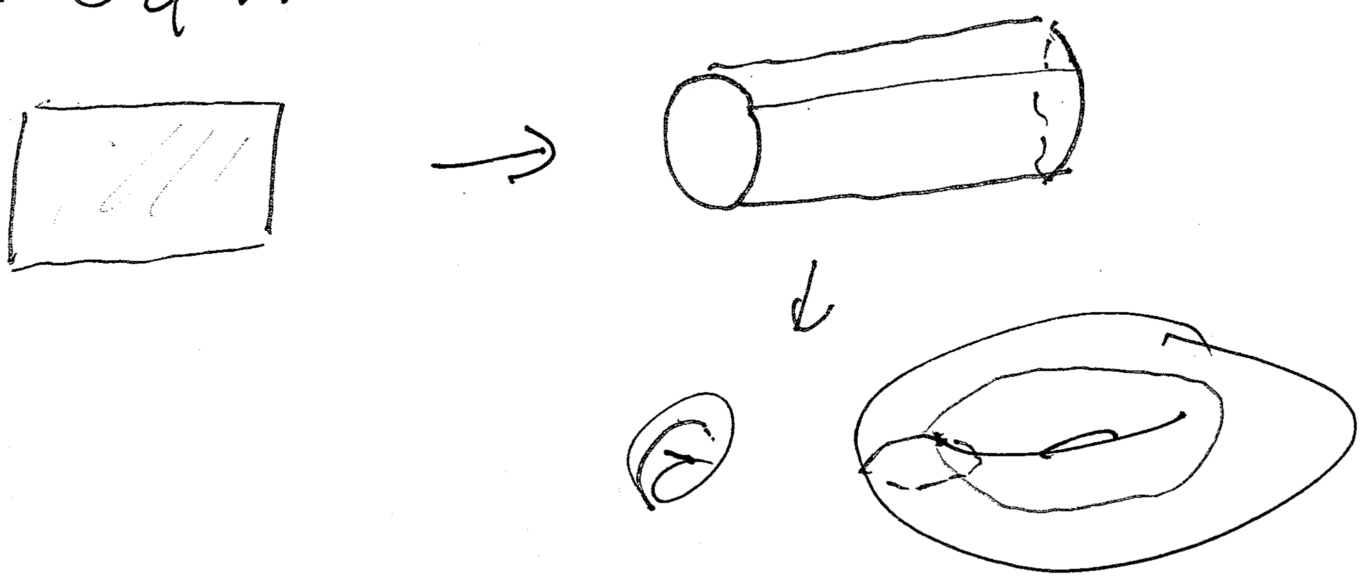
using the theorem $p: \mathbb{R} \to S^1$, $p': \mathbb{R} \to S^1$

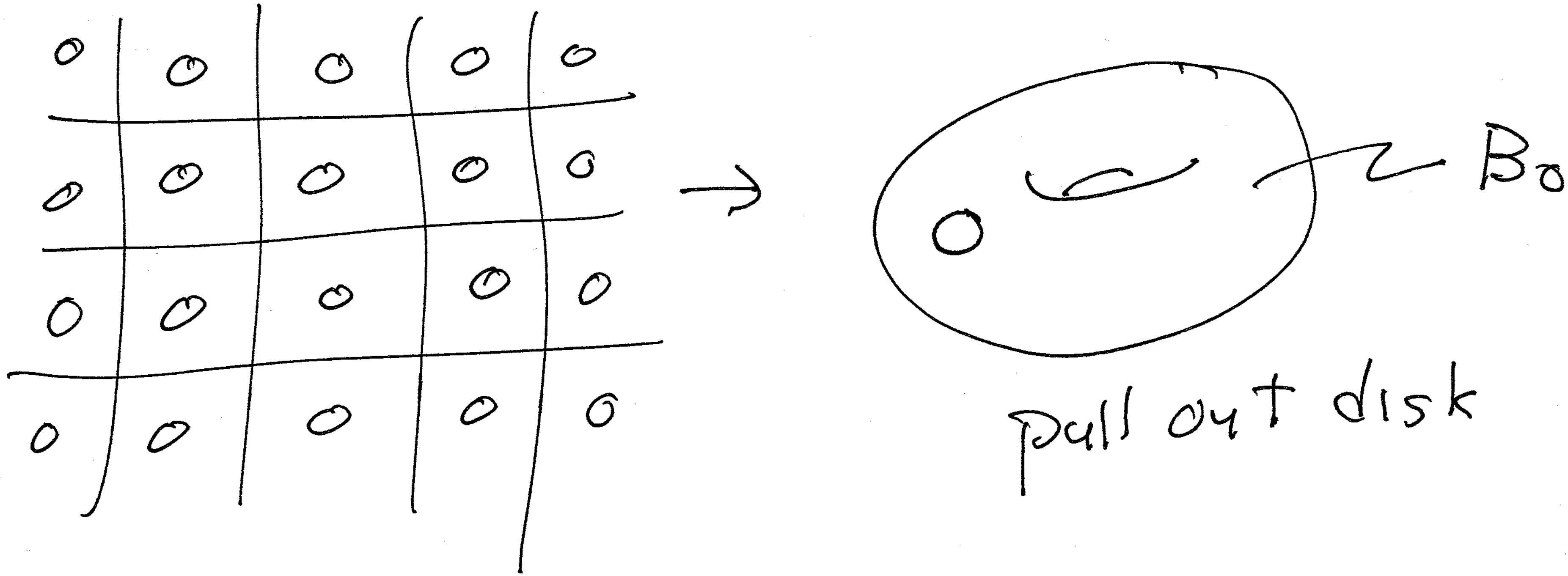
as above then

$(p \times p'): \mathbb{R}^2 \longrightarrow \mathbb{T}^2$

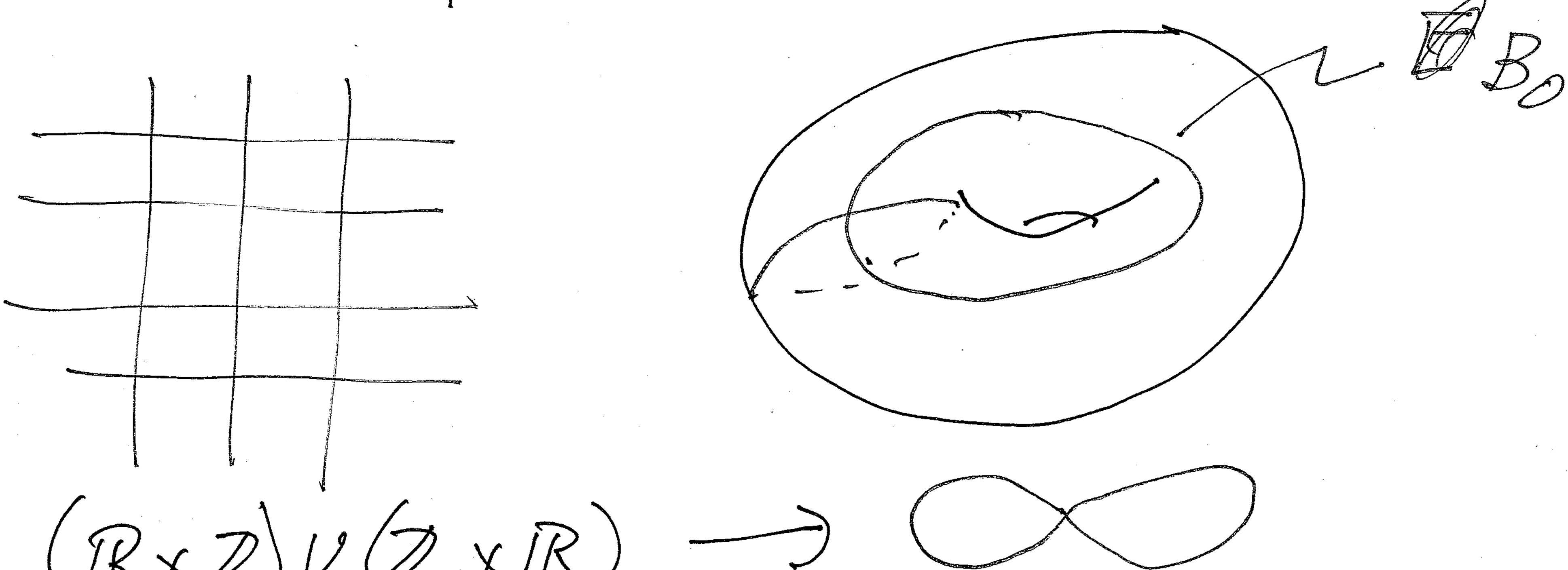


each square





pull out disk



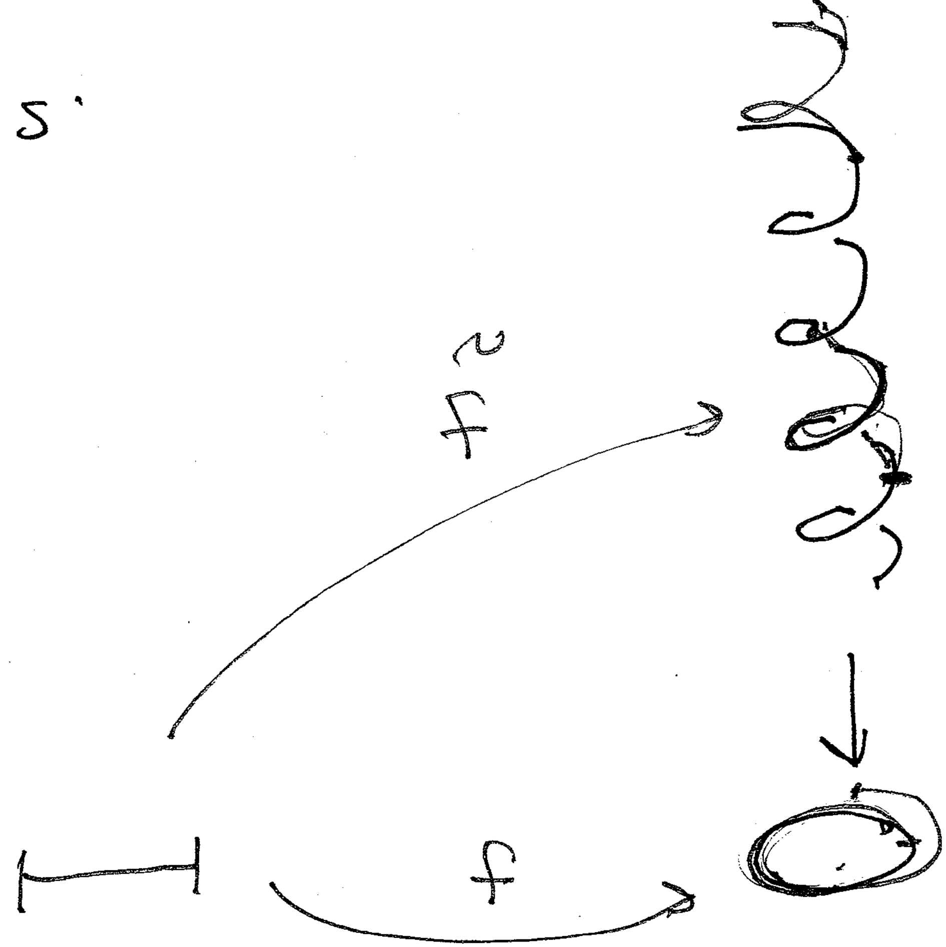
$(\mathbb{R}\times\mathbb{Z})\cup(\mathbb{Z}\times\mathbb{R})$

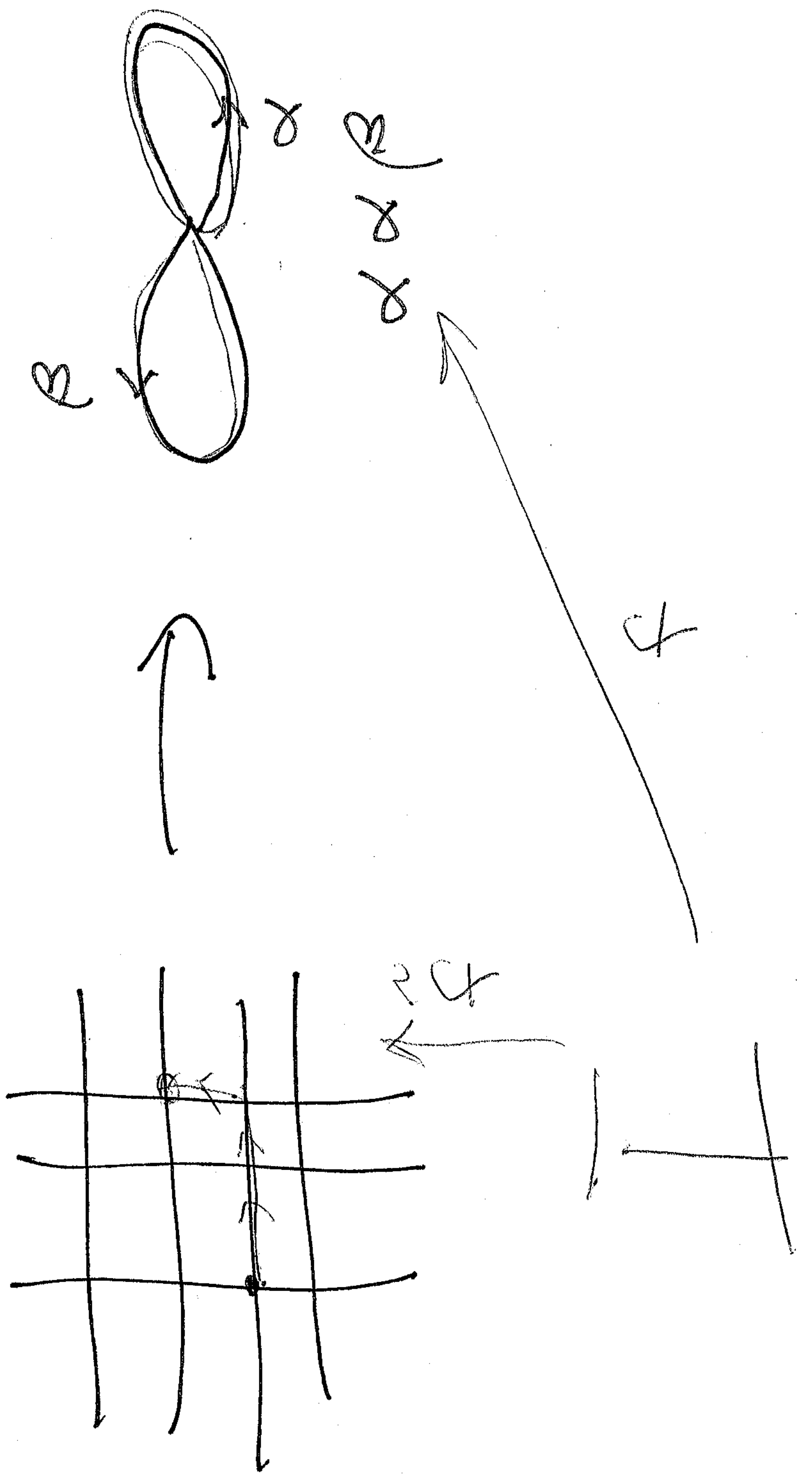
DEF:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B \end{tikzcd}$$

Given $f: X \to B$ if $\exists\, \tilde{f}: X \to E$ with $p \circ \tilde{f} = f$ then $\tilde{f}$ is called a lift of f.

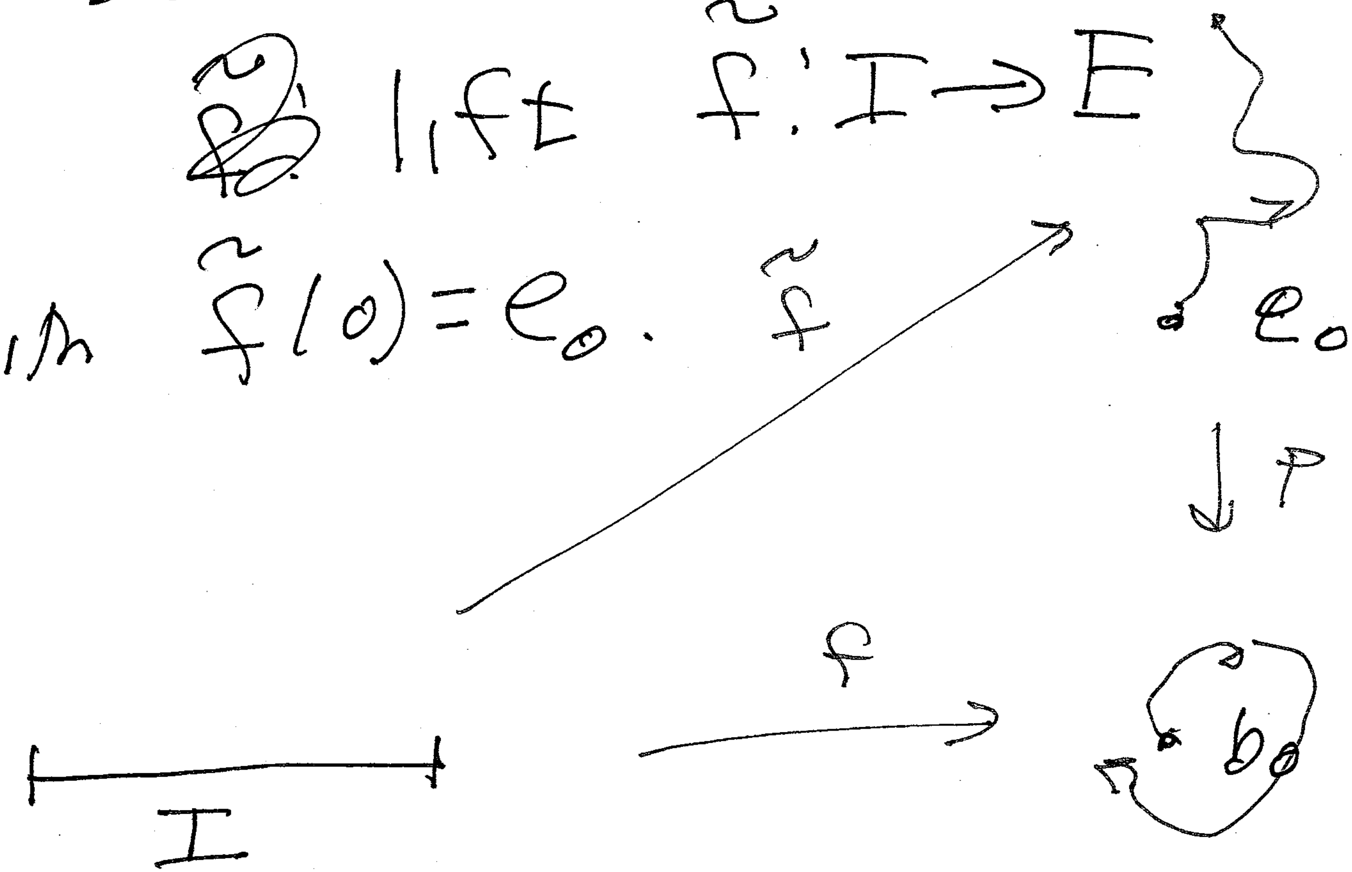
Examples:



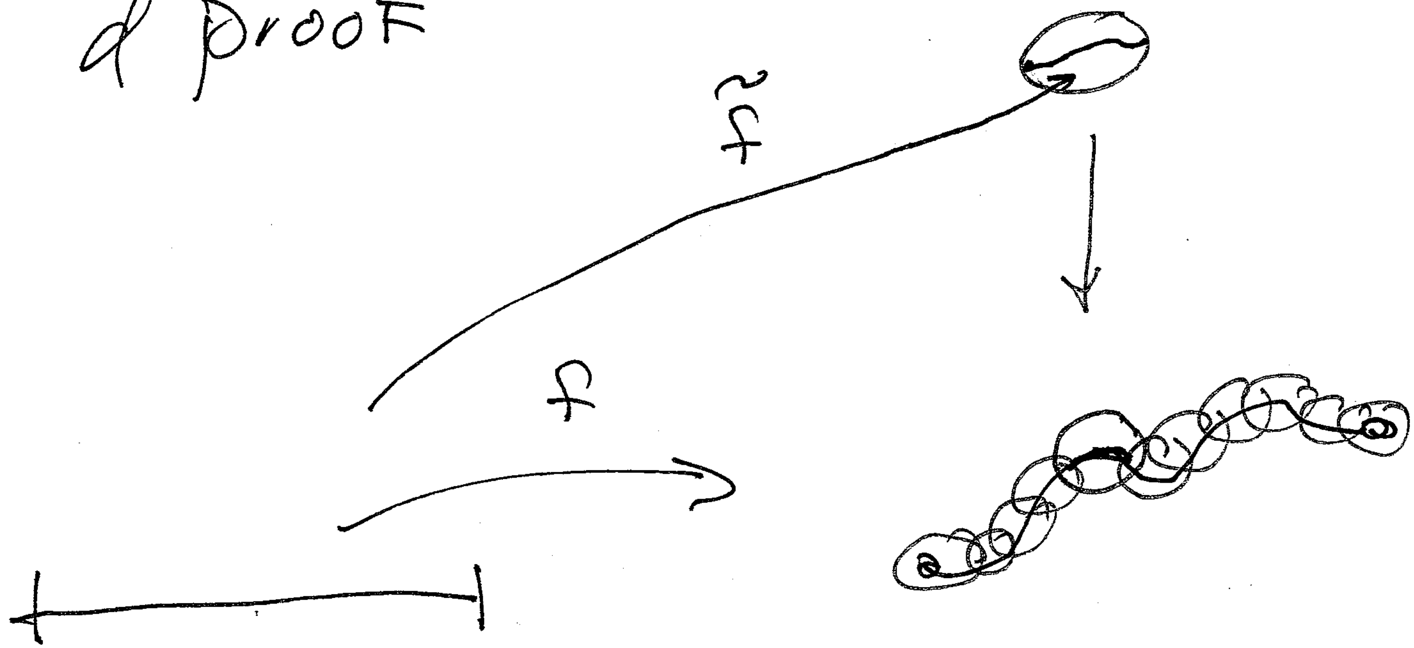


γ
β
γ
γ
β
α
2α

Lemma (Path Lifting): $p: E \to B$ c.s. $e_0 \in E$ and $p(e_0) = b_0$. If $f: I \to B$ with $f(0) = b_0 \Longrightarrow$ $\exists !$ lift $\tilde{f}: I \to E$ with $\tilde{f}(0) = e_0$.



Picture of proof



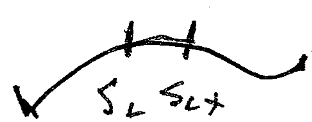
Proof Let $\{U_\gamma\}$ be a cover of B by open sets that are evenly covered. This induces a cover of $f(I)$ which cpt.

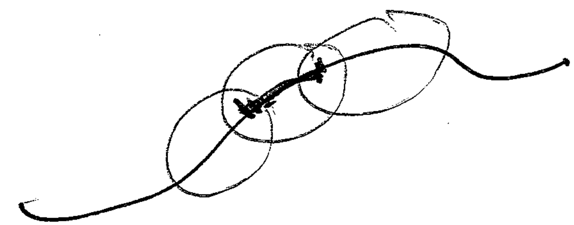
Using the Lebesgue number Lemma (27.5)

We may subdivide $[0,1]$ as

$$0 = s_0 < s_1 < \dots < s_n = 1 \quad \text{with}$$

$$f([s_i, s_{i+1}]) \subseteq U_{\lambda_i} \text{ for some } \lambda_i$$





We define $\tilde{f}$ inductively.

Let $\tilde{f}(0) = e_0$

Assume $\tilde{f}(s)$ is defined for $0 \le s \le s_i$ we define $\tilde{f}$ on $[s_i, s_{i+1}]$ as follows:

Now $f([s_i, s_{i+1}]) \subseteq U_{x_0}$

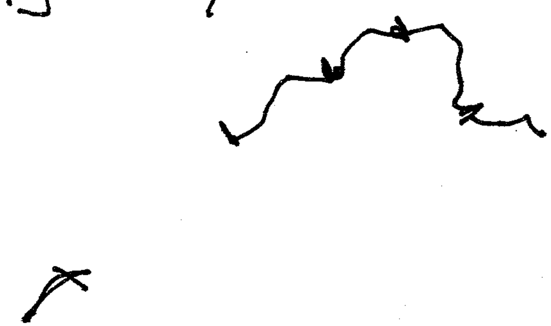
and $p^{-1}(U_{x_0}) = \coprod V_\alpha$ and $\tilde{f}(s_i)$ is in one of them say V_{α_0}. Define $\tilde{f}$ on $[s_i, s_{i+1}]$ as $\tilde{f}(s) = \left(p|_{V_{\alpha_0}}\right)^{-1} \circ f(s)$

$P|_{V_0}$ is a homeo so

$\tilde{f}$ is cont. on $[s_i, s_{i+1}]$

and thus on $[0, s_{i+1}]$ by the

pasting Lemma.



Uniqueness is similar. ∎