__Set up__

$$\begin{array}{c}(E, e_0)\\ \downarrow p\\ (B, b_0)\end{array}$$

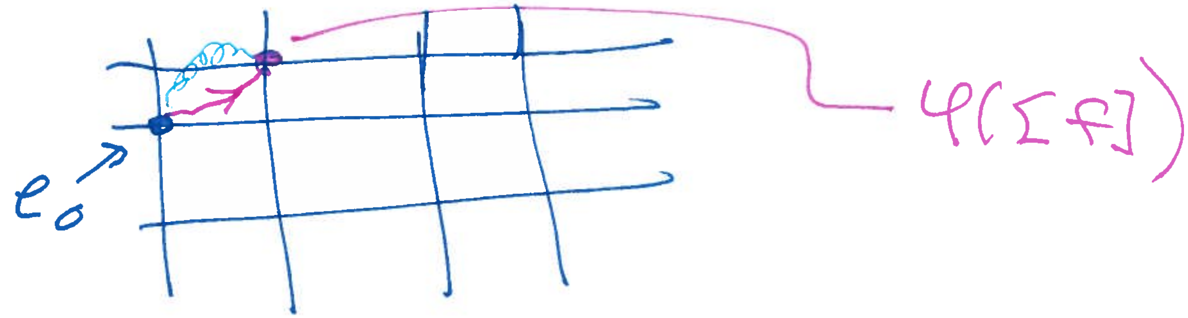
Define $\varphi : \pi_1(B, b_0) \to \pi^{-1}(b_0)$

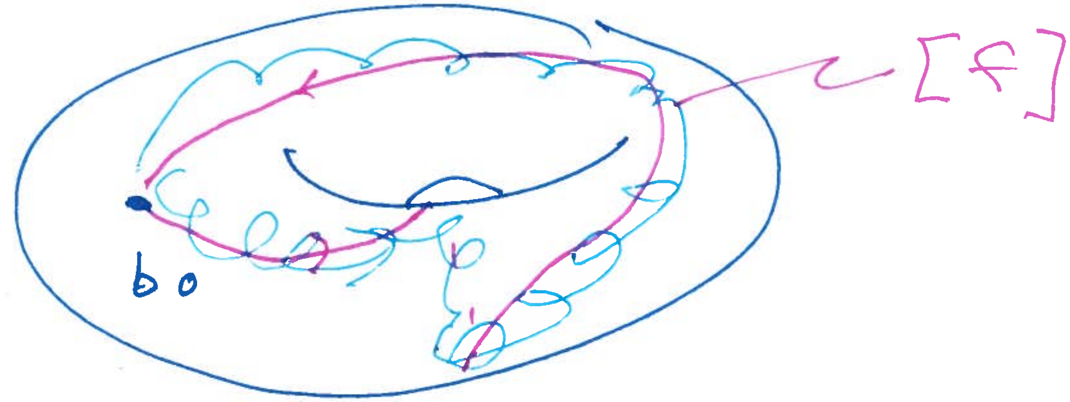
via $[f] \to \tilde{f}(1)$ where

$\tilde{f}$ is the unique lift of f with

$\cancel{\tilde{f}(e_0) =}$ $\tilde{f}(0) = e_0$

Last time - well defined.

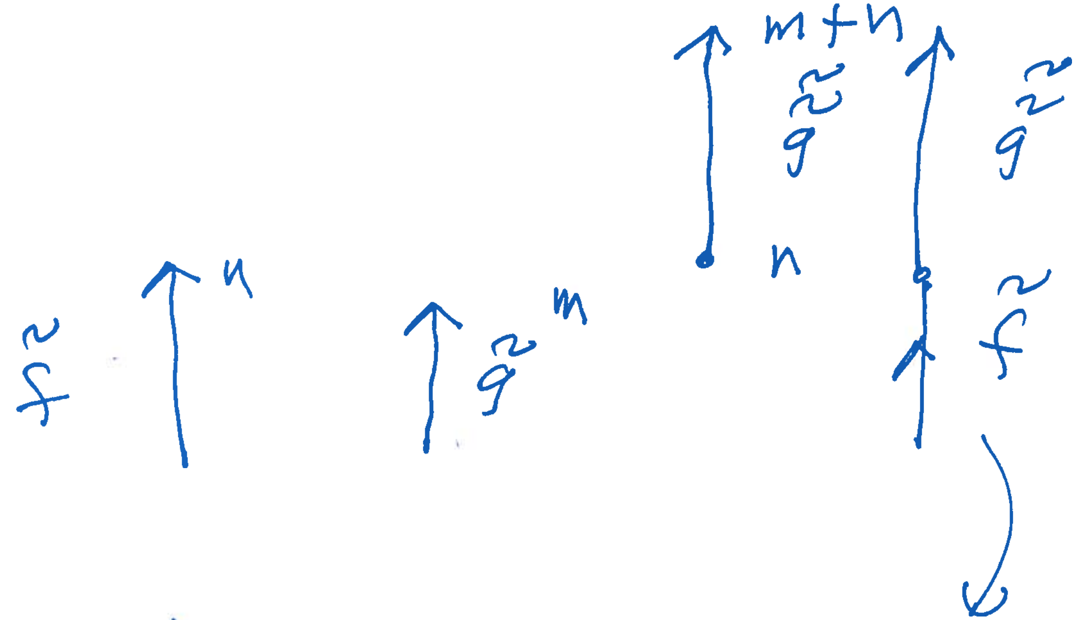




Lemma: E is path conn + simply conn $\Rightarrow$ φ is bijective

Theorem: $\varphi : \pi_1(S^1, 1) \to p^{-1}(1) = \mathbb{Z}$ is a group homomorphism and thus a isomorphism since $\mathbb{R}$ is simply conn and bijective homomorphisms are isomorphisms.

Proof $\varphi([f] * [g]) = \varphi([f]) + \varphi([g])$ is to be proved.



$\tilde{f}$
n
$\tilde{g}$
m
$m+n$
$\tilde{\tilde{g}}$
n
$\tilde{\tilde{g}}$
$\tilde{\tilde{f}}$

$f * g$

Pick $[f], [g] \in \pi_1(S^1, 1)$

and their unique lifts $\tilde{f}, \tilde{g}$

~~satisfy $\varphi[\tilde{f}] =$~~ with $\tilde{f}(0) = 0 = \tilde{g}(0)$

and $\varphi[\tilde{f}] = n$ or $\tilde{f}(1) = n$

$\varphi[\tilde{g}] = m$ or $\tilde{g}(1) = m$

Let $\tilde{\tilde{g}}$ be the path $\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$

So $\tilde{\tilde{g}}(0) = n$ $\quad \tilde{\tilde{g}}(1) = m + n$.

$\tilde{f} * \tilde{\tilde{g}}$ is a path from

0 to $m+n$.

$$p(\tilde{\tilde{g}}) = p(n+\tilde{g}) = p(\tilde{g}) = g$$

$$p(n+x) = p(x) \qquad p(x) = e^{2\pi i x}$$

$$\Rightarrow p(\tilde{\tilde{f}} * \tilde{\tilde{g}}) = f * g.$$

$$\phi([f] * [g]) = m + n$$

$$= \phi([f]) + \phi([g]).$$

Similarly $\pi_1(T^2, (0,0)) \cong \mathbb{Z}^2$

$\pi_1(T^n, (0,0)) \cong \mathbb{Z}^n$

Homework

Applications

Reminder: Set theory

$\alpha \circ \beta = id$

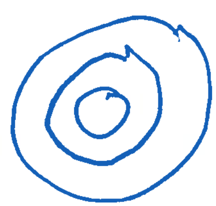
$\Rightarrow$ α is onto

β is 1-1.

DEF: $A \subseteq X$, $i_A : A \to X$ is the inclusion $i_A(a) = a$.

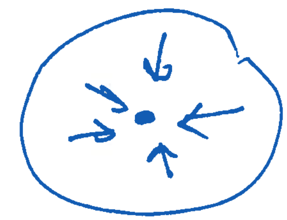
DEF: $A \subseteq X$ a retraction of X onto A is a continuous $r : X \to A$ s.th. $r|_A = \mathrm{id}$ and $r \circ i_A = \mathrm{id}_A$

Example (1) $X = \{(r,\theta) : \frac{1}{2} \le r \le 3/2\}$ in polar coord.



$A = S^1$ $r : X \to A$ $(r,\theta) \mapsto (1,\theta)$

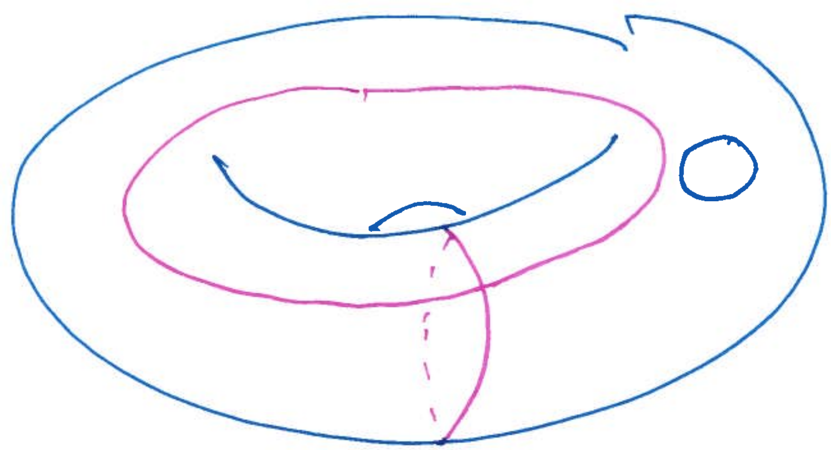
(2) $X = D^2$, $A = \vec{0}$ $D^2 = \{ z \in \mathbb{C} : |z| \le 1 \}$

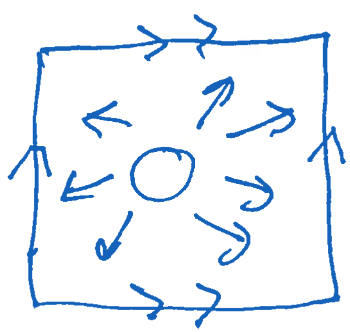


$= \{ (r, \theta) : r \le 1 \}$.

$R : (r, \theta) \longrightarrow \vec{0}$

(3)





A is called a <u>retract</u> of X

if $\exists$ a <u>retraction</u>

$r : X \to A$.

Th'm: There is no retraction
$D^2 \to S^1$

Lemma: If A is a retract of
X, $a \in A \Rightarrow (i_A)_*: \pi_1(A,a) \to \pi_1(X,a)$
is injective.

Proof $r \circ i_A = id_A$ since r is retract
$r_* \circ (i_A)_* = (id_A)_*$
by prelim remark, $(i_A)_*$ is injective.

Proof of Theorem:

Assume $\exists\, r : \cancel{\partial} D^2 \to S^1$

$$\Rightarrow \quad (i_{S^1})_* : \pi_1(S^1, 1) \to \pi_1(D^2, 1)$$

ie an injective homomorph

$$\mathbb{Z} \to \{e\}$$

a contradiction. $\boxtimes$

__Lemma__: $h: S^1 \to X$ continuous. TFAE

(1) h is null homotopic (h is homotopic to a constant map)

(2) h extends to a continuous $k: D^2 \to X$.

(3) h_* is a trivial homomorphism

$h_*: \pi_1(S^1, 1) \to \pi_1(X, h(1))$

Also say it can be extended

Proof next time