2.8

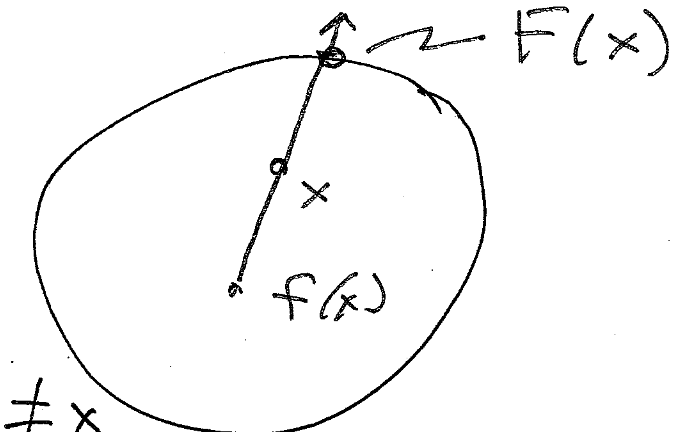
Proof of Th'm at end of last time is in the book.

Brouwer Fixed point Th'm in dimension 2.

Th'm: $f : D^2 \to D^2$ $(D^2 = \{ \vec{x} \in \mathbb{R}^2 : \|\vec{x}\| = 1 \})$ is continuous function $\Rightarrow$ $\exists\, x_0$ with $f(x_0) = x_0$.

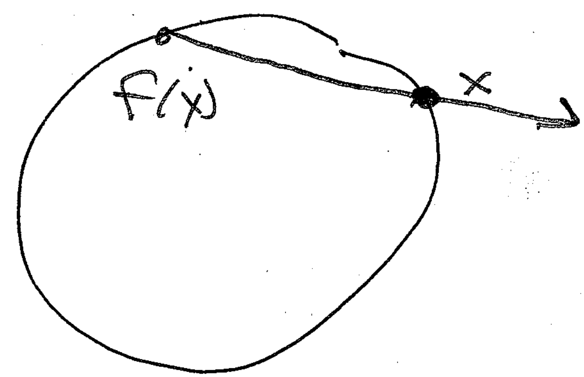
Proof: Assume BWOC that $f(x) \neq x$ $\forall x \in D^2$

Define $F: D^2 \to S^1$ as the point where the ray from $f(x)$ to x hits S^1



F Defined since $f(x) \neq x$ and is continuous

and. $F(x) = x$ for $x \in S^1$

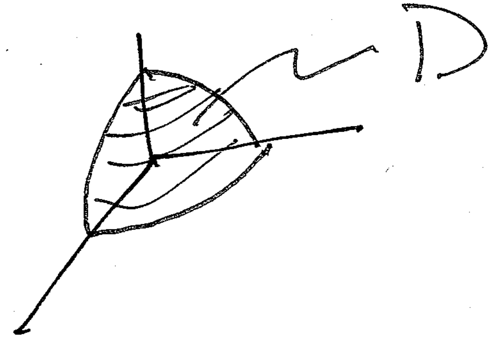


So F is a refraction $D^2 \to S^1$, a contradiction. ∎

Perron-Fröbenius – special case.

Th'm: M is a real 3×3 matrix with $M_{ij} > 0 \ \forall i,j$. $\Rightarrow$ M has an eigenvalue $\lambda > 0$ whose eigenvector $\vec{v}$ satisfies $v_i \geq 0$.

Proof

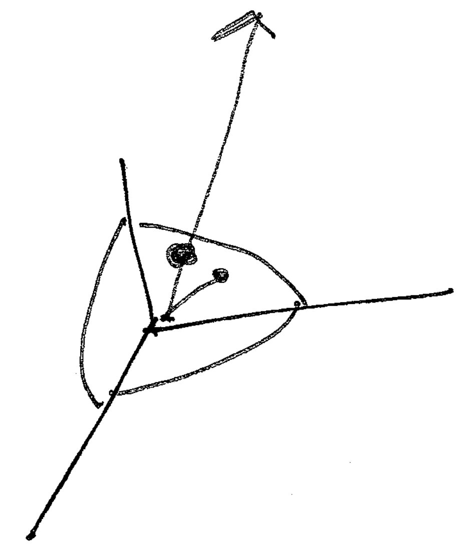


$$D = \{\vec{x} \in \mathbb{R}^3 : \|\vec{x}\| = 1, \text{ all } x_i \geq 0\}$$

so D is homeomorphic to D^2

Define $f : D \to D$ via

$$\cancel{f(\vec{x}) = \frac{M(\vec{x})}{\|Mx\|}}$$



defined since $M > 0$

$$f(\vec{x}) = \frac{M(\vec{x})}{\|M(\vec{x})\|}$$

is continuous. By Brouwer

$\exists$ $\vec{x}' \in D$ with $f(\vec{x}') = \vec{x}'$

So $M(\vec{x}') = \|M(\vec{x}')\| \vec{x}'$

$\lambda = \|M(\vec{x}')\|$ $\quad \vec{v} = \vec{x}'$ $\quad \blacksquare$

<u>Remark</u>: In the proper metric, f is a contraction (Hilbert metric)

$\Rightarrow \exists! \ \vec{v}$ with $\vec{v} > 0$

Another application: The Fundamental Theorem of Algebra. (<u>See Book</u>)

Borsak-Ulam Theorem.

$f : S^n \to \mathbb{R}^n$ is continuous

$\Rightarrow \exists \ x \in S^n$ with $f(x) = f(-x)$

$n=1$ last time semester HW

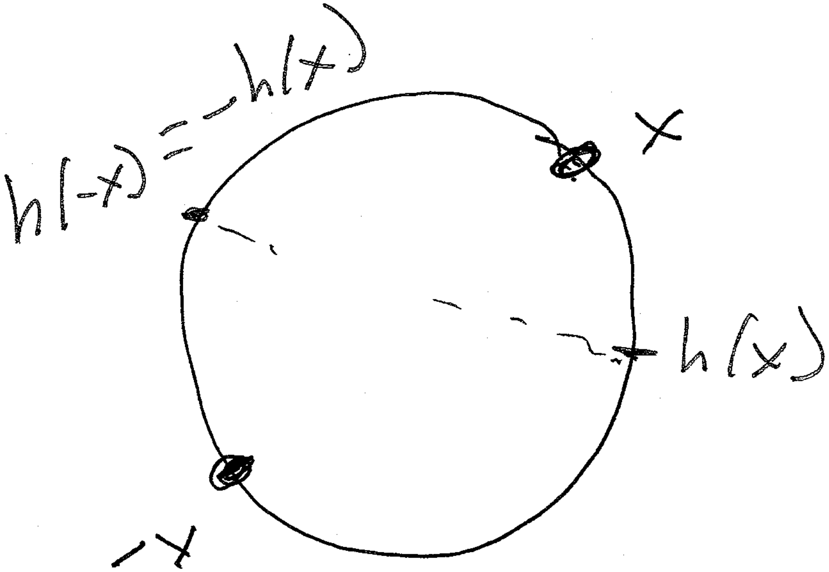
$n=2$ today.

$\boxed{\text{Application}}$ On the earth $\exists$ a pair of antipodal points with same temperature and humidity.

__DEF__ $x \in S^n \Rightarrow -x \in S^n$ is the antipodal point
$\subseteq \mathbb{R}^{n+1}$ $\quad S^n = \{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$

__DEF__: $\cancel{h} : S^n \to S^n$ is antipodal preserving (or odd) is

$$h(-x) = -h(x)$$



examps

(1) $h(x) = x + \alpha \mod 2\pi$
rigid rotation

(2) $h(z) = z^2$

(3) $h(z) = z^3$

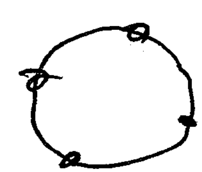
$z \in S^1 \subseteq \mathbb{C}$

(2) $h(-z) = (-z)^2 = z^2$ even not odd

(3) $h(-z) = (-z)^3 = -z^3$ odd

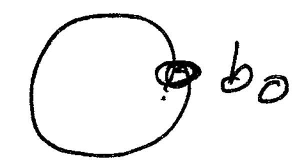
antipode preserving.

<u>intuition</u>: $h: S^1 \to S^1$ odd can't be null homotopic

 not 

Theorem: $h: S^1 \to S^1$ is antipodal preserving continuous $\Rightarrow$ h is not null homotopic.

Proof. Step 0: get a basepoint preserving map. Let $b_0 = 1$



Let $\rho: S^1 \to S^1$ rigid rotation that ~~that~~ takes $h(b_0)$ to b_0. Since ρ is odd then so $\rho \circ h$ is also.

Further If $H: h \simeq \text{const} \Rightarrow \rho \circ H \circ \rho \circ h \simeq \text{const}$

So we make ~~as~~ work with $\rho \circ h$ with $\rho \circ h(b_0) = b_0$.

So WLOG $h(b_0) = b_0$.

Step 1: Auxillary function: $g(z) = z^2$

$g : S^1 \to S^1$ and g is even and

$g^{-1}(z) =$ pair of antipodal points

So $g \circ h$ is even. Claim

$\exists$ continuous k with

$$\begin{array}{ccc} S^1 & \xrightarrow{h} & S^1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ S^1 & \xrightarrow{k} & S^1 \end{array}$$

$$\begin{tikzcd} S^1 \arrow[r, "h"] \arrow[d, "q"'] & S^1 \arrow[d, "q"] \\ S^1 \arrow[r, "k"] & S^1 \end{tikzcd}$$

Informally $k = q h q^{-1}$

$q^{-1}(z)$ is a pair of antipodal pts.

$h q^{-1}(z)$ is also a pair of antipodal pts
since h is antipodal preserving.

so $q h q^{-1}(z)$ is a single pt since
q sends antipodal points to the
same point.

continuity follows locally

and. $k(1) = q h q^{-1}(1)$

$= q h \{-1, 1\}$

$= q \{-1, 1\} = 1$

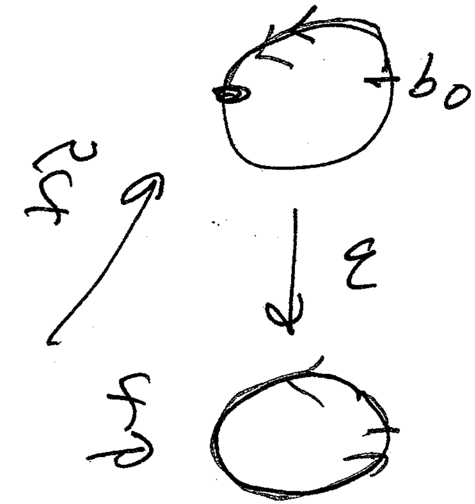
$h(1) = 1$

so $h(-1) = -1$

So $k(b_0) = b_0$.

<u>Step 2</u>: We show $k_* : \pi_1(S^1, b_0) \to \pi_1(S^1, b_0)$ is nontrivial.

For this $q: S^1 \to S^1$ in fact is a covering space and if $\tilde{f}\partial$ is a path in S^1 with $b_0 \to -b_0$



$\Rightarrow f = q \circ \tilde{f}$ is notriv in $\pi_1(S^1, b_0)$

since $\tilde{f}$ doesn't start and end at b_0

Thus $k_*[f] = [k \circ q \circ \tilde{f}]$

$= [q \circ h \circ \tilde{f}]$

nontrivial since $h \circ \tilde{f}$ is a path is S^1 from b_0 to $-b_0$.

So $\exists$ element in $\pi_1(S^1, b_0)$ whose image under K_* is not zero

So K_* is nontrivial.

Step 3 next time