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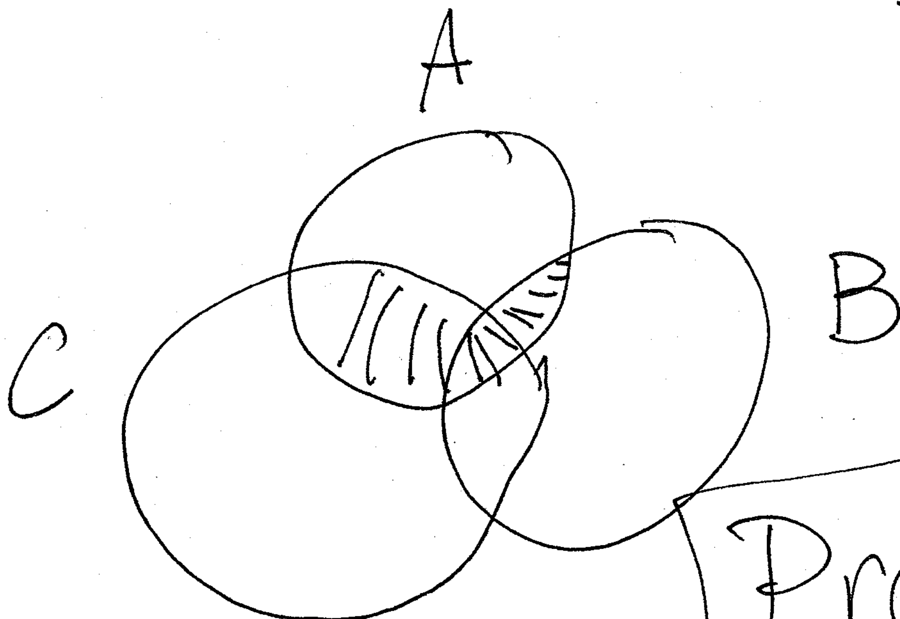
# Algebra of sets

operations $\cup, \cap, -, \times$

union intersection Set differency Cartesian product

$Set \longrightarrow Set$

<u>Formula</u> $\underbrace{A \cap (B \cup C)}^{LHS}$ $= \overbrace{(A \cap B) \cup (A \cap C)}^{RHS}$




LHS = left hand side
RHS = right . . .

$\boxed{\text{Prove}}$ LHS $\subseteq$ RHS
RHS $\subseteq$ LHS

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

**Proof:** $x \in LHS$

$\Rightarrow$ $x$ is in $B$ or $C$ or both

and $x$ is in $A$

if $x \in B \Rightarrow x \in A \cap B$

$x \in C \Rightarrow x \in A \cap C$

$\Rightarrow x \in (A \cap B) \cup (A \cap C) = RHS$

$x \in RHS$

$\Rightarrow$ $x$ is is either in both $A$ and $B$ or in both $A$ and $C$

$\Rightarrow$ $x$ is in $A$ and in $B \cup C$

$\Rightarrow$ $x \in A \cap (B \cup C) = \underline{LHS}$.

## Other rules

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- $A - (B \cup C) = (A - B) \cap (A - C)$
- $A - (B \cap C) = (A - B) \cup (A - C)$

De Morgan's Law

Extend $\cup$ and $\cap$ to larger collections of sets

$$\mathcal{A} = \{ A_1, A_2, \ldots, A_{1000} \}$$

each $A_i$ a set

$$\bigcup_{i=1}^{1000} A_i = \{ x : x \in A_i \text{ for some } i \}$$

$$\bigcap_{i=1}^{1000} A_i = \{ x : x \in A_i \text{ for } \cancel{\text{som}} \text{ all } i \}$$

Index set

$\Lambda = \{1, 2, \ldots, 1000\}$

$$\bigcup_{i \in \Lambda} A_i = \bigcup_{i=1}^{1000} A_i$$

index set can be infinite

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$\Lambda$ is an index set

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for some } x \in \Lambda\}$$

## Index Sets

$$\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$$

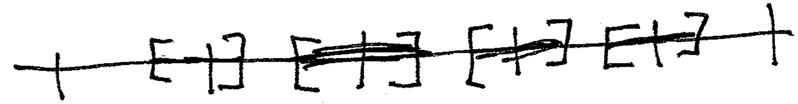
$$\mathbb{N} = \{ 0, 1, 2, \ldots \}$$

$$\mathbb{Z}_+ = \{ 1, 2, 3, \ldots \}$$

<u>example</u>

$$A_n = [n - 1/4, n + 1/4] \quad \text{(closed interval)}$$

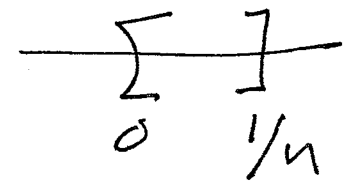
$$X = \bigcup_{n \in \mathbb{Z}} A_n$$



$$\bigcap_{n \in \mathbb{Z}} A_n = \emptyset = \{\ \} \quad \text{empty set}$$

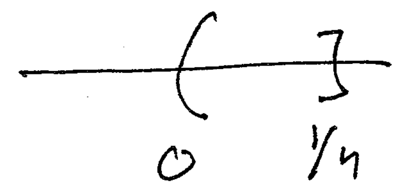
$$\{\ \}$$

$$B_n = [0, 1/n] \qquad n \in \mathbb{Z}_+$$



$$\bigcap_{n \in \mathbb{Z}_+} B_n = \{0\}$$

$$C_n = (0, 1/n]$$



$$\bigcap_{n \in \mathbb{Z}_+} C_n = \emptyset$$

Cartescian Product

$$A \times B = \{ (a, b) : a \in A \quad b \in B \}$$

$$A_i \quad i = 1, \ldots, N$$

$$A_1 \times \ldots \times A_N = \{ (a_1, a_2, \ldots, a_N) : a_i \in A_i \}$$

$$\prod_{i=1}^{N} A_i = \bigtimes_{i=1}^{N} A_i$$

$$\mathbb{R} \times \ldots \times \mathbb{R}^n = \mathbb{R}^n = \prod_{i=1}^{n} \mathbb{R}$$

$\mathbb{Z}^n \subseteq \mathbb{R}^n$ ↳ a integer lattice

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functions and relations

<u>Informally</u> $f : C \longrightarrow D$

is a rule that assigns each $c \in C$ and $d \in D$ called $d = f(c)$.

$C$ domain

$D$ = range, target space

not every $d \in D$ has to be $f(c)$ for some $c$.

Image$(f) = \{d \in D : \exists c \in C$

exists

with $f(c) = d\}$

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$f : \mathbb{R} \to \mathbb{R} \quad f(x) = x^2$

Image$(f) = [0, \infty)$.

range$(f) = \mathbb{R}$

formally

$f: C \to D$ is a subset

$y \subseteq C \times D$ such $\forall c \in C$

for all

exists unique $d$ with

$(c, d) \in y$ write $d = f(c)$.

eg $f(x) = x^2$, $(1,1)$, $(2,4)$

$(-2, 4)$, ... are in $y$.

So a function is a

<u>triple</u> <u>$f, C, D$</u>

eg// Let $\tilde{\mathbb{R}}_+ = \{x \in \mathbb{R} : x \geq 0\}$

$f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

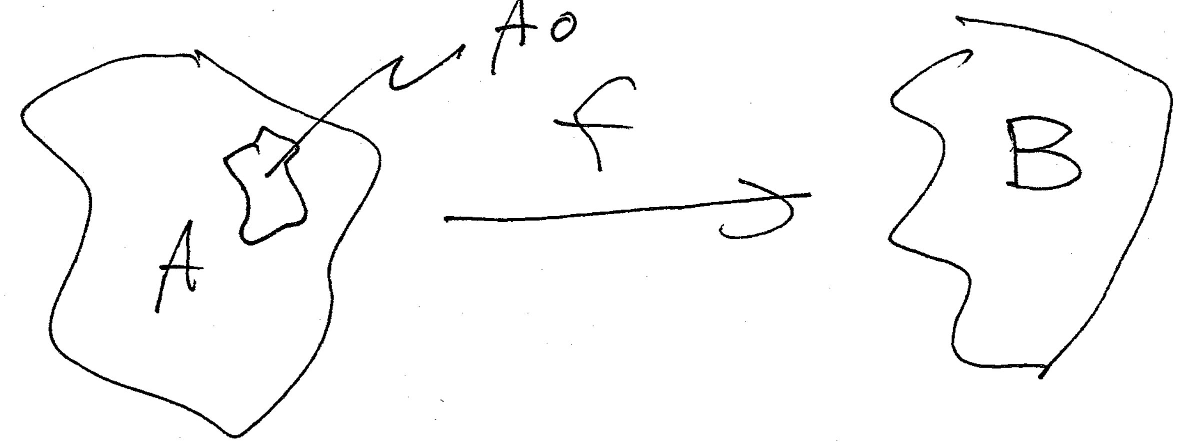
$f : \mathbb{R} \to \tilde{\mathbb{R}}_+$ " "

$f : \tilde{\mathbb{R}}_+ \to \mathbb{R}$ " "

$f : \tilde{\mathbb{R}}_+ \to \tilde{\mathbb{R}}_+$ " "

all different triples so all different functions.

Restrictions




$A_0 \subseteq A$ $\quad f|_{A_0}$ = restriction of $f$ ~~A~~ to $A_0$

so $f|_{A_0} : A_0 \to B$