Subspace Topologies

$Y \subseteq X$, the subspace topology on Y given $\mathcal{T}$ on X is

$$\mathcal{T}_{|Y} = \{ U \cap Y : U \in \mathcal{T} \}$$

<u>examples:</u> $(0,1)$ in $\mathbb{R}_S$ - examples

$(-1, 1/2) \cap (0,1) = (0, 1/2)$ open

$(\frac{1}{4}, 1/2)$ is open

$Y = [0,1] \rightarrow$ open set exampl

$(-1, 1/2) \cap [0,1] = [0, 1/2)$

open in $\mathcal{T}_{|Y}$ not in $\mathbb{R}_S$

Sometimes $U \cap Y$ is called open relative to Y

<u>Lemma</u>: the subspace topology is a topology.

$\boxed{\text{Formulas in proof}}$

open in $\mathcal{T}$

$$\bigcap_{i=1}^{n}(U_i \cap Y) = \left(\bigcap_{i=1}^{n} U_i\right) \cap Y$$

$$\bigcup_{\lambda \in \Lambda}(U_\lambda \cap Y) = \left(\bigcup_{\lambda \in \Lambda} U_\lambda\right) \cap Y$$

__expected Lemma__ If $\mathcal{B}$ is a base for $(X, \mathcal{T})$ and $Y \subseteq X$

$$\Rightarrow \mathcal{B}_{|Y} := \{ B \cap Y : B \in \mathcal{B} \}$$

is a base for $\mathcal{T}_{|Y}$.

$\boxed{\text{Proof in book}}$

example $Y = [0,1)$, $\mathcal{B} = \{(a,b)\}$ for $\mathbb{R}_S$

$$\mathcal{B}_{|Y} = \{ [0,a) \ a \le 1, \ (a,b) \ 0 \le a \le b \le 1 \}$$

is a base $\mathcal{T}_{|Y}$

When are relative open sets open in the ambient space.

<u>**Lemma:**</u> $Y \subseteq X$ and $Y \in \mathcal{T}$

$\Rightarrow$ U open Y implies U open X.

<u>**Proof**</u> Since U is open in Y

$\Rightarrow$ $U = Y \cap V$ for $V \in \mathcal{T}_X$

so U is intersection of two ambient open sets and so is ambient open. $\blacksquare$

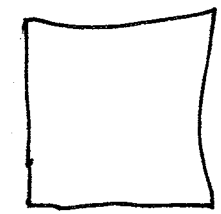
examples

$$[0,1) \wedge \{2\} = Y$$

$\Rightarrow$ $\{2\}$ is open in Y

not in $\mathbb{R}$.

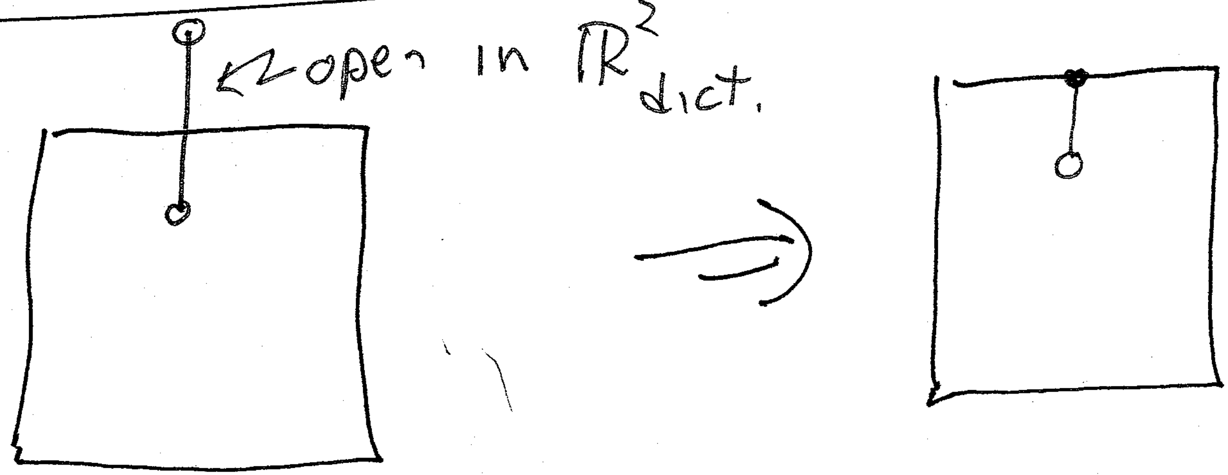
example $S = [0,1]^2$.



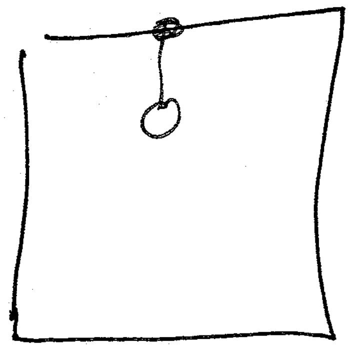
Two topologies

(1) dictionary topology on S.

(2) Restriction of the dictionary topology on $\mathbb{R}^2$ to S.



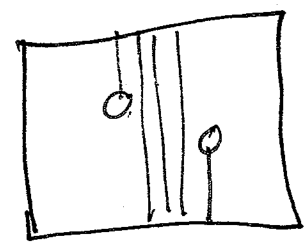
not open in (1)



any open set in (2)

~~looks~~

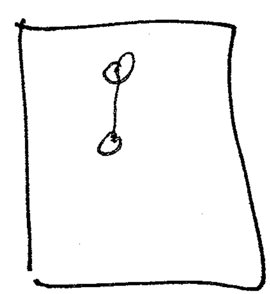
looks like

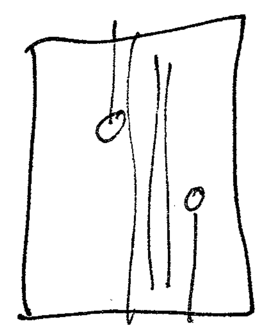


or it is not the union of base

elements of (1)

which are open intervals
in the order structure





$\boxed{\text{Closed sets}}$

DEF: Given $(X, \mathcal{T})$

$A \subseteq X$ is closed $\iff$ $X - A$ is open.

NOTE: $X - A = A^c$, also $X \setminus A = X - A$

setminus

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eg// ① $\mathbb{R}_s$ $\mathbb{R} - [0,1] = (-\infty, 0) \cup (0, \infty)$

$\Rightarrow [0,1]$ is closed.

(2) In $\mathbb{R}_s$

$$\mathbb{R} - (0,1] = (-\infty,0] \cup (1,\infty)$$

not open so $(0,1]$ not closed.

(3) In $\mathbb{R}_\ell$ which has base $\{[a,b)\}$

$$\mathbb{R} - [0,1) = \underbrace{(-\infty,0)}_{\text{open}} \cup \underbrace{[1,\infty)}_{\text{open}}$$

$\Rightarrow [0,1)$ is closed in $\mathbb{R}_\ell$ and also open.

5 Finite topology:

U is open in $X \iff U^c$ is finite

$\Rightarrow$ closed sets are exactly the finite sets

6 In $\mathbb{R}_s$, $\mathbb{R} - \{a\} =$

$(-\infty, a) \cup (a, \infty)$

So pts are closed.

Using de Morgan's law the collection of closed sets satisfies

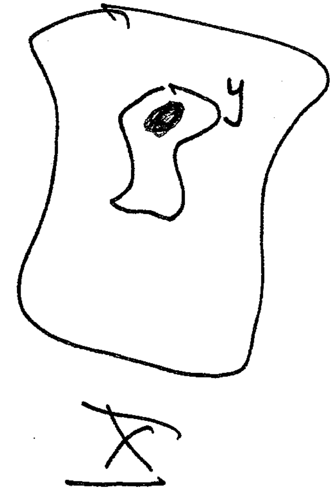
(1) X, ϕ are closed

(2) closed under finite unions.

(3) closed under arbitrary intersections

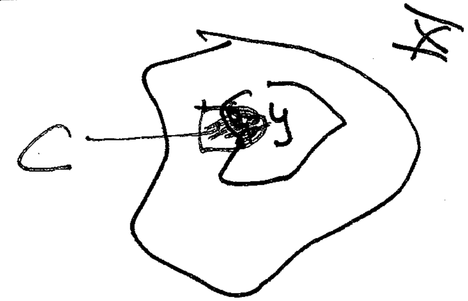
DEF $Y \subseteq X$, $A \subseteq Y$

is closed in Y or relatively closed in Y, if and only if $Y - A$ is open $\mathcal{T}|_Y$

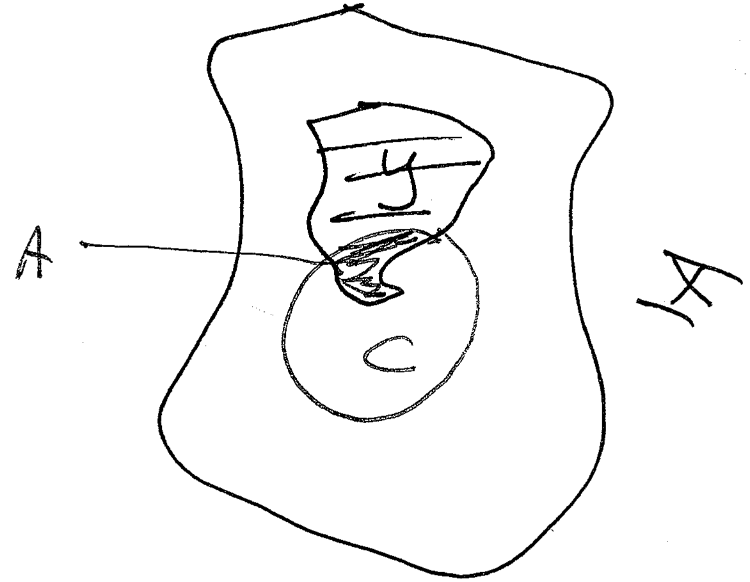


Theorem $Y \subseteq X$

A is closed in Y $\iff$ $A = Y \cap C$ with C closed in X.



Proof ($\Leftarrow$)



Assume $A = Y \cap C$
with C closed in X.

So $X - C$ is open X
then $(X - C) \cap Y$ is open in Y
$\mathcal{T}_{|Y}$ / but $\boxed{(X - C) \cap Y = Y - A}$
So $Y - A$ is open in Y
$\Rightarrow A$ is closed in Y

($\Rightarrow$) Conversely, say A is closed in Y

$\Rightarrow$ $Y - A$ is open in Y

$\Rightarrow$ $\exists$ U open in X with

$Y - A = U \cap Y$

and so $A = Y \cap (X - U)$

with $X - U$ closed in X $\blacksquare$

Closure and interior of a set

Intuition $Cl((0,1)) = [0,1]$

$Cl([0,1] \cap \mathbb{Q}) = [0,1]$

DEF Given $A \subseteq (X, \mathcal{T}) \Rightarrow$

The closure of $A = \bar{A} = Cl(A)$ is the intersection of all closed sets containing A.

The interior of A, ~~Int~~ $\mathrm{Int}(A) = \mathring{A}$

is the union of all open sets

contained inside A.

$$\mathrm{Int}([0,1] \cup \{2\}) = (0,1) \quad \text{in } \mathbb{R}_S$$