DEF $A \subseteq (X, \mathcal{T})$ x is a limit point of A, if U is a nbd of x

$$\Rightarrow \quad U \cap (A - \{x\}) \neq \emptyset$$

or $x \in \mathrm{cl}(A - \{x\})$

example $\{\frac{1}{n} : n \in \mathbb{Z}_+\} = K$

only limit pt is $\{0\}$.

$$\overline{K} = K \cup \{0\}.$$

Theorem $\overline{A} = A \cup A'$ (A' = all limit pts of A)

($\supseteq$) $A \subseteq \bar{A}$ by def

So assume $x \in A'$

$\Rightarrow$ by def any nhd U of x hits

$A - \{x\} \Rightarrow x \in \bar{A}$ by a previous th'm

so $A' \subseteq \bar{A}$.

($\subseteq$) $x \in \bar{A}$ $x \in A$ done.

So assume $x \notin A$. By a th'm every

nhd U of x has $U \cap A \neq \emptyset$, but $x \notin A$

so $U \cap A - \{x\} \neq \emptyset \Rightarrow x \in A'$

Theorem: If $(X, \mathcal{T})$ is T_1 (finite sets are closed), then

$x \in A' \iff$ all nbhds of x contain infinitely many points of A.

Proof ($\Leftarrow$) Triv ($\Rightarrow$) BWOC (By way of contradiction) assume $x \in A'$ and some nbhd U of x is such that $U \cap A$ is finite.

$U \cap A$ is finite

Let $U \cap (A - \{x\}) = \{x_1, \ldots, x_n\}$.

By T_1, $U - \{x_1, \ldots, x_n\}$ open

~~Thus $U \cap A \{x_1, \ldots, x_n\}$ is a nhd of x that misses $A - \{x\}$. So x is not a limit pt. ∎~~

Thus $U \cap (X - \{x_1, \ldots, x_n\})$ is a nhd of x that misses $A - \{x\}$

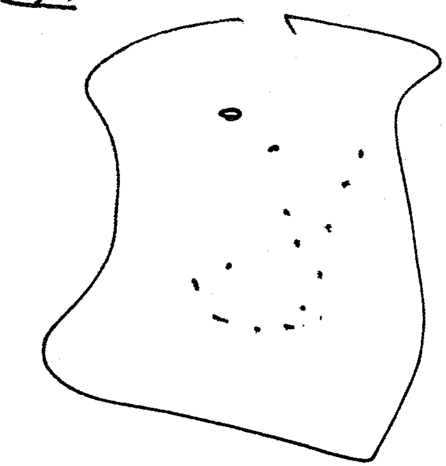
$\Rightarrow x \notin A'$.

$\boxed{\text{Sequences}}$ $\{x_n\}_{n \in \mathbb{Z}_+} \subseteq X$

formally is a map $\phi : \mathbb{Z}_+ \to X$

and we write $x_n = \phi(n)$

or $\{x_n\}$ is countable collection of points in X.

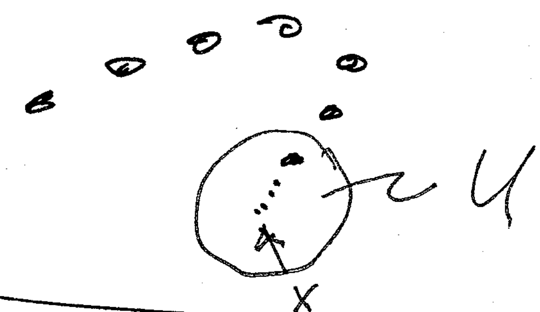


$x_n \to x$ or $\lim_{n\to\infty} x_n = x$ or

$\{x_n\}$ converges x if.

For all nbd U of x, $\exists N$ so that $n \geq N \Rightarrow x_n \in U$.

i.e. all nbds of U contain all but finitely many points from the sequence.



calculus in $\mathbb{R}_S$

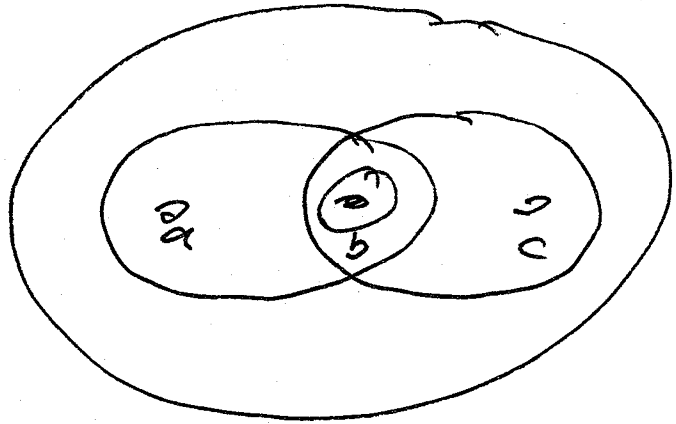
given $\varepsilon > 0$, $U = (x-\varepsilon, x+\varepsilon)$

$\Rightarrow \exists N, n \geq N \Rightarrow x_n \in U$

$= |x - x_n| < \varepsilon$

eg// (1) $x_n = \frac{1}{n}$, $\frac{1}{n} \to 0$

(2)



$x_n = b$ for all n.

$x_n \to b$

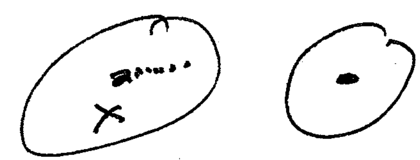
every nhd of a contains b. $x_n \to a$ also

and $x_n \to c$ also.

(3) $x_n = \{-1\}^n$ doesn't converge.

__Theorem__ $(X, \mathcal{T})$ is HD $\Rightarrow$ a sequence converges to at most one point.

__Proof__ Assume $x_n \to x$ and pick $y \neq x$. Since $\mathcal{T}$ is HD $\exists$ open U, V $x \in U$, $y \in V$ $U \cap V = \emptyset$



Since $x_n \to x$, U contains all but finitely elements of $\{x_n\}$

so $x_n \not\to y$.

doesn't converge.

Continuous functions

DEF: $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$
is continuous if $\forall V \in \mathcal{T}_Y$

$f^{-1}(V) \in \mathcal{T}_X$

"inverse images of open sets are open"

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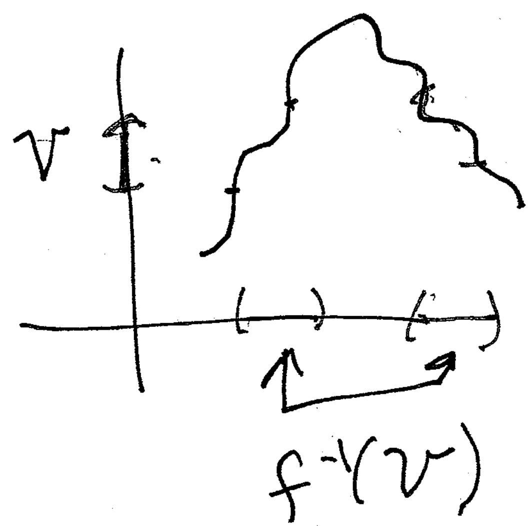
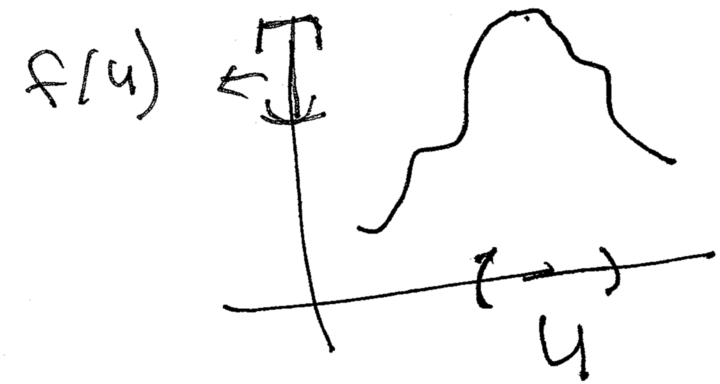
note: (1) depends on f and the topology on the range and domain.

(2) If $\mathcal{B}$ is a base for $\mathcal{T}_Y$ suffices to have $f^{-1}(B) \in \mathcal{T}_X$ $\forall B \in \mathcal{B}$

(3) If $\mathcal{S}$ is a subase for $\mathcal{T}_Y$
$\Rightarrow$ suffices to check $f^{-1}(S) \in \mathcal{T}_X$
for all $S \in \mathcal{S}$.

Pictures of graphs for $f: \mathbb{R}_S \to \mathbb{R}_S$





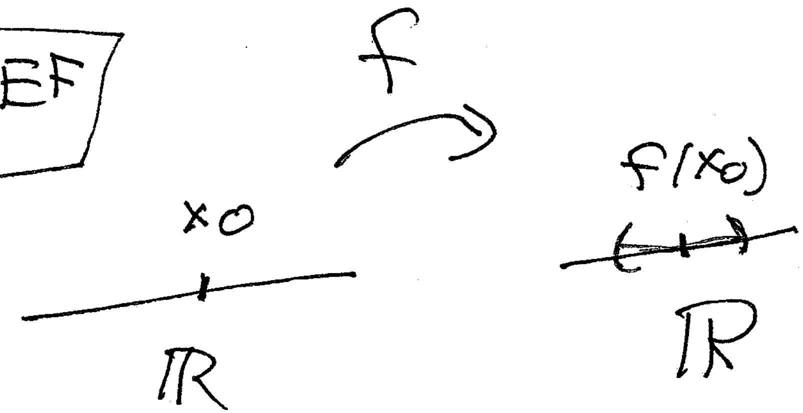
$f(U)$ not open.

Connecting to Calculus def for $f : \mathbb{R}_S \to \mathbb{R}_S$

CALC DEF $\forall x_0, \forall \varepsilon > 0 \; \exists \delta > 0$ so that

$$|x_0 - x| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

$\boxed{\text{TOP DEF} \Rightarrow \text{CALC DEF}}$

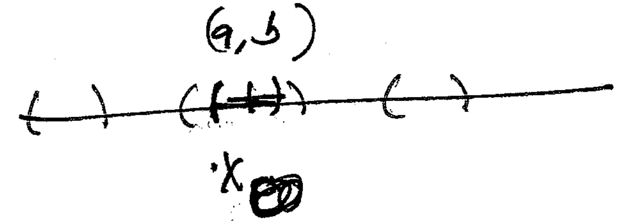


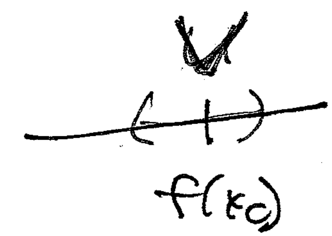
Given x_0, $\varepsilon > 0$

form $V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

open in $\mathbb{R}_S$

By Top def, $f^{-1}(V)$ is open

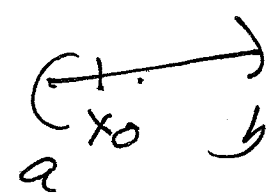




and $x_0 \in f^{-1}(V)$

so $\exists$ basedelement (a,b) with

$x_0 \in (a,b) \subseteq f^{-1}(V)$



Let $\delta = \min(x_0 - a, b - x_0)$

Then $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(V)$

$\Rightarrow f((x_0 - \delta, x_0 + \delta)) \subseteq V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$

or $|x_0 - x| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

$\boxed{\text{CALC} \Rightarrow \text{TOP also}}$

Examples

$id : \mathbb{R}_\ell \to \mathbb{R}_S$

check base elements

$(id)^{-1}((a,b)) = (a,b)$ open $\mathbb{R}_\ell$

so cont

$id : \mathbb{R}_S \to \mathbb{R}_\ell$

check base elements (a,b), and $[a,b)$

$(id)^{-1}([a,b)) = [a,b)$ not open in $\mathbb{R}_S$ so not cont.

In general

$$\mathcal{T}_1 \supseteq \mathcal{T}_2 \text{ on } X \iff$$

$$\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2) \text{ is}$$

continuous.