DEF: $f : (X, T_X) \to (Y, T_Y)$ is continuous if

$$f^{-1}(U) \in T_X$$

often the topologies are assumed and one just says $f : X \to Y$ is cont.
Theorem: \( f: X \to Y \) TFAE

1. \( f \) is cont.
2. \( \forall A \subseteq X, \ f(\overline{A}) \subseteq \overline{f(A)} \)
3. \( \forall \text{closed } B \subseteq Y, \ f^{-1}(B) \text{ closed in } X \)
4. \( \forall x \in X, \ \forall \text{nbd } U \ni x \) \( f(U) \cap f(U) = \emptyset \)

\[ f(a) \]
Proof Logic

\[ (\forall \Rightarrow \exists) \quad x \in A \quad \text{let } u \]
be an element \( f(x) \) by cont. \( f^{-1}(u) \)

open in \( X \) and \( x \in f^{-1}(u) \)

Since \( x \in A \), \( \exists y \in f^{-1}(u) \cap A \)

\[ \exists y \in f^{-1}(u) \cap A \]

So \( f(y) \in f(A) \)

\[ \Rightarrow f(A) \subseteq f(A) \quad \text{for all } x \in A \]

\[ \Rightarrow f(A) \subseteq f(A) \]
(2) \Rightarrow (3) \quad \text{Assume } B \text{ is closed in } Y. \quad A = f^{-1}(B). \quad \text{We show } \overline{A} = A

which implies \ A \text{ is closed.}

f(A) = f(f^{-1}(B)) \subseteq B

If \ x \in \overline{A} \Rightarrow

f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B

\overline{f(A)} \subseteq \overline{B} = B

(2) \quad \overline{f^{-1}(B)} = A

So \ x \in \overline{f^{-1}(B)} = A

\overline{A} = A \quad \text{but } A \subseteq \overline{A} \text{ always}

\overline{A} = A. \quad \text{so } \overline{A} = A.
(3) \Rightarrow (1) \quad \text{Assume } V \text{ open in } Y
\[ B = Y - V \text{ is closed in } Y \]
\[ f^{-1}(B) = f^{-1}(Y) - f^{-1}(V) \]
\[ = \overline{X} - f^{-1}(V) \]
\[ f^{-1}(B) \text{ closed by hyp} \]
\[ \Rightarrow f^{-1}(V) \text{ open.} \]
(1) \Rightarrow (4) If \( x \in \overline{X} \), \( V \) is a
neighborhood of \( f(x) \) then \( U = f^{-1}(V) \) is
open in \( X \) by hyp. and \( x \in U \)
and \( f(x) \in V \). So \( U \) is the
desired open set.

(4) \Rightarrow (1) Assume \( V \) is open set
Pick \( x \in f^{-1}(V) \) so \( f(x) \in V \)
By hyp \( \exists \) \( U_x \) of \( x \) with
\( f(U_x) \subseteq V \) so \( U_x \subseteq f^{-1}(V) \)
Do this for every \( x \in f^{-1}(A) \)

\[
\text{Den } f^{-1}(A) = \bigcup_{x \in f^{-1}(A)} U_x
\]

So \( f^{-1}(A) \) is the union of open sets and so is open \( \Box \)
When are \((X, \mathcal{G}_x) \sim (Y, \mathcal{G}_y)\) identical in terms of topological properties?

\[
\begin{align*}
E_g (0, 1) \quad \text{and} \quad \mathbb{R}
\end{align*}
\]

**DEF** \(f : (X, \mathcal{G}_x) \to (Y, \mathcal{G}_y)\) is a **homeomorphism** if

1. bijection
2. \(f\) is cont
3. \(f^{-1}\) is cont.
Alternative characterization

(A) $f$ is bijective and
$U$ is open $\iff f(U)$ is open.
So open sets are in 1-1 correspondence.

So $f$ is like a nice change of coordinates

(B) $f$ is bijection and
$f(\text{open}) = \text{open}$ and
$f^{-1}(\text{open}) = \text{open}$. 
Erlangen Program

Topology is the study of properties invariant under homeomorphism.

Example: $f: \mathbb{R}_5 \to \mathbb{R}_5$

$f(x) = 3x + 1$
$g(x) = \frac{1}{3} (y - 1)$

homeo $\mathbb{R}_5 \to \mathbb{R}_5$. 
$f : (-1,1) \rightarrow \mathbb{R}$

$(-1,1)$ has subspace from $\mathbb{R}$

$f(x) = \frac{x}{1-x^2}$

$f' > 0$, monotone↑

So order preserving

as $1$ is $f^{-1}$

So it preserves the order topology

which is also standard top.

So is homeo.
f doesn't preserve lengths

doesn't preserve geometry

And here's an example.
$S^1 = \text{Unit circle } = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$

With $S^1 \subseteq \mathbb{R}^2$, $\rho : S^1 \to \mathbb{R}$

$f(\theta) = (\cos 2\pi \theta, \sin 2\pi \theta)$

$f$ is cont., bijective.
Is it a homeomorphism?

\[ \sigma_{0,1/4} \text{ open in } [0,1) \]

\[ \sigma_{0,1/4} = \sigma_{0,1} \cap (-1/4, 1/4) \]

\[ f(\sigma_{0,1/4}) \text{ is not open in } S! \]

So \( f \) is cont, bijective but not homeo
To be open in $\mathbb{S}$

$\text{Top}$

$\forall x \in B \exists U \text{ open in } \mathbb{R}^2 \text{ with } x \in U$

$\forall B \in \mathbb{B} \text{ and } \forall S \subseteq B$

$\forall S \subseteq B$

not true for $f(0)$