DEF: $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous

if $\forall\, V \in \mathcal{T}_Y \;\; f^{-1}(V) \in \mathcal{T}_X$.

Often the topologies are assumed and one just says $f : X \to Y$ is cont.

Theorem: $f: X \to Y$ TFAE

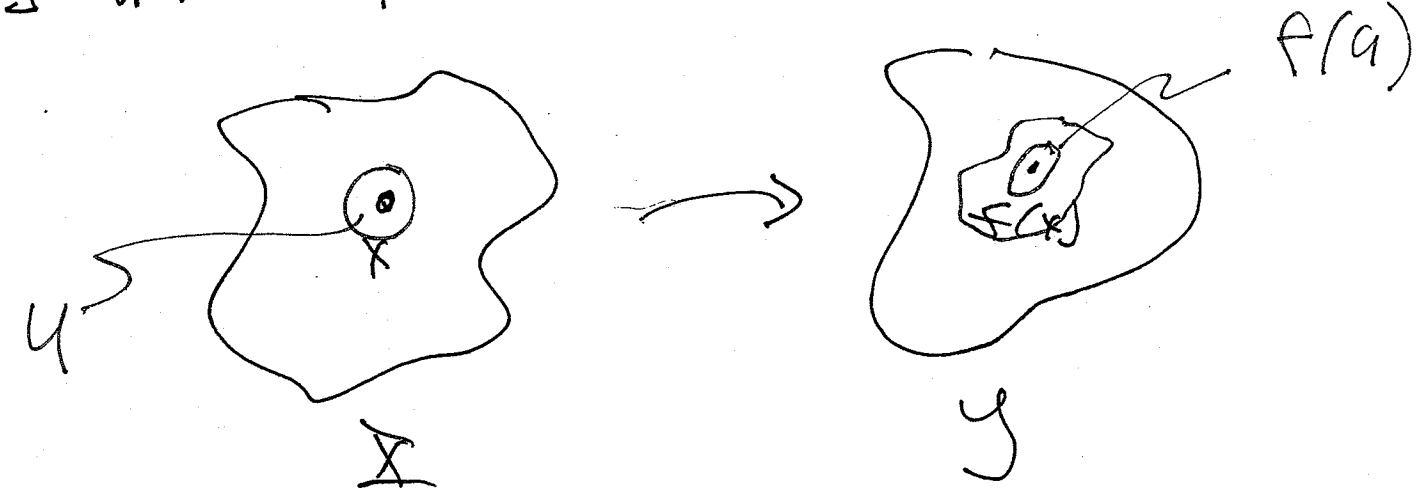
(1) f is cont.

(2) $\forall A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$

(3) $\forall$ closed $B \subseteq Y$, $f^{-1}(B)$ closed in X

(4) $\forall x \in X$, $\forall$ nbd V of $f(x)$

$\exists$ nbd U of x with $f(U) \subseteq V$



Proof Logic

$$\begin{tikzcd} 1 \arrow[r, bend left] \arrow[rr, bend left] & 2 \arrow[r, bend left] & 3 \arrow[l, bend left] & 4 \arrow[lll, bend left] \end{tikzcd}$$

(1) $\Rightarrow$ (2) $x \in \bar{A}$ let V be a nbhd $f(x)$ by cont. $f^{-1}(V)$ open in X and $x \in f^{-1}(V)$

Since $x \in \bar{A}$, $\exists\, y \in f^{-1}(V) \cap A$

$f(y) \in V \cap f(A)$

So $f(x) \in \overline{f(A)}$ for all $x \in \bar{A}$

$\Rightarrow f(\bar{A}) \in \overline{f(A)}$

(2) $\Rightarrow$ (3) Assume B is closed in Y. $A = f^{-1}(B)$. We show $\bar{A} = A$ which implies A is closed.

$$f(A) = f(f^{-1}(B)) \subseteq B$$

If $x \in \bar{A} \Rightarrow$

$$f(x) \in f(\bar{A}) \underset{(2)}{\subseteq} \overline{f(A)} \subseteq \bar{B} = B$$

$$\boxed{\begin{array}{c} C \subseteq D \\ \Rightarrow \bar{C} \subseteq \bar{D} \end{array}}$$

So $x \in f^{-1}(B) = A$

$\bar{A} \subseteq A$ but $A \subseteq \bar{A}$ always so $\bar{A} = A$.

(3) $\Rightarrow$ (1) Assume V open in Y

$B = Y - V$ is closed in Y

$$f^{-1}(B) = f^{-1}(Y) - f^{-1}(V)$$

$$= X - f^{-1}(V)$$

$f^{-1}(B)$ closed by hyp

$\Rightarrow f^{-1}(V)$ open.

(1) $\Rightarrow$ (4) If $x \in X$, V is a nbd of $f(x)$ then $U = f^{-1}(V)$ is open in X by hyp. and $x \in U$ and $f(U) \subseteq V$. So U is the desired open set.

(4) $\Rightarrow$ (1) Assume V is open set

Pick $x \in f^{-1}(V)$ so $f(x) \in V$

By hyp $\exists$ nbd U_x of x with $f(U_x) \subseteq V$ so $U_x \subseteq f^{-1}(V)$

Do this for every $x \in f^{-1}(V)$

Then $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$

So $f^{-1}(V)$ is the union of open sets and so is open $\blacksquare$

When are $(X, \mathcal{T}_X)$ $(Y, \mathcal{T}_Y)$ identical in terms of topological properties.

$\boxed{\text{Eg } (0,1) \text{ and } \mathbb{R}}$

<u>DEF</u> $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a <u>homeomorphism</u> is

(1) bijection

(2) f is cont

(3) f^{-1} is cont.

Alternative characterization

(A) f is bijective and

U is open $\Leftrightarrow$ $f(U)$ is open.

So open sets are in 1-1 correspondance

$\boxed{\text{So } f \text{ is like a nice change of coordinates}}$

(B) f is bijection and

$f(\text{open}) = \text{open}$ $f^{-1}(\text{open}) = \text{open}$.

Erlangen program

~~But~~ Topology is the study of properties invariant under homeomorphism

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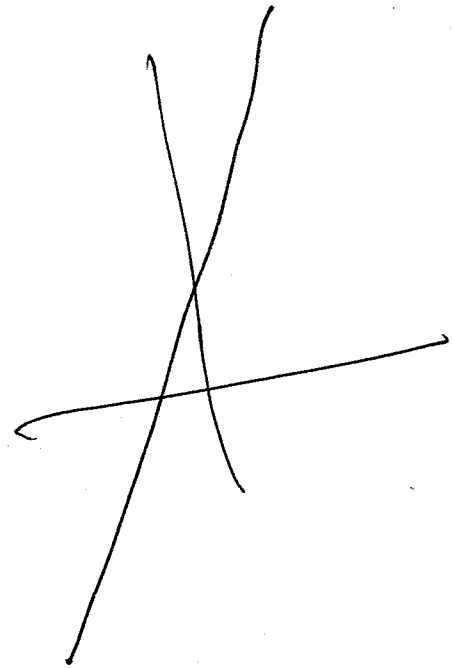
<u>examples</u> $f : \mathbb{R}_s \to \mathbb{R}_s$

$$f(x) = 3x+1$$
$$g(x) = \frac{1}{3}(y-1)$$

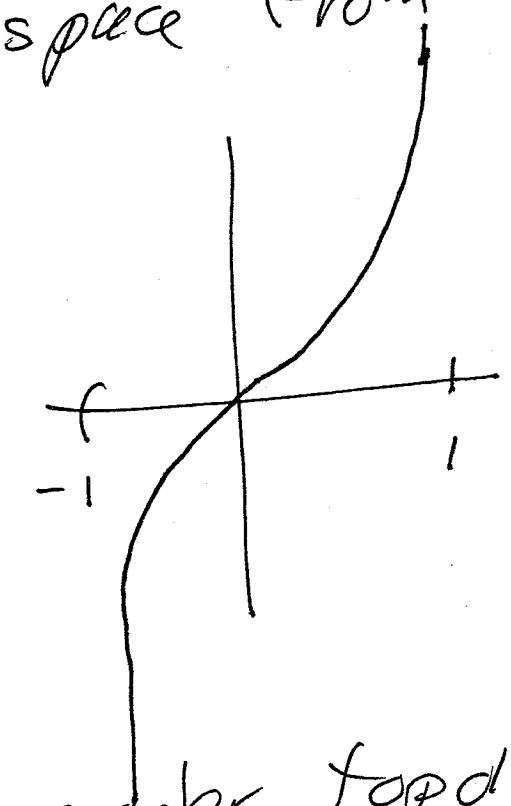


homeo $\mathbb{R}_s \to \mathbb{R}_s$.

$f : (-1,1) \to \mathbb{R}_S$

$(-1,1)$ has subspace from $\mathbb{R}_S$

$$f(x) = \frac{x}{1-x^2}$$



$f' > 0$, monotone $\uparrow$

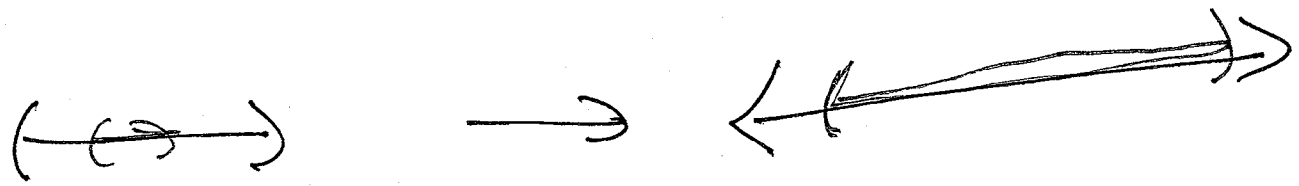
so order preserving

as is f^{-1}

so it preserves the order topology which is also standard top.

so is <u>homeo</u>.

f doesn't preserve lengths



doesn't preserve geometry

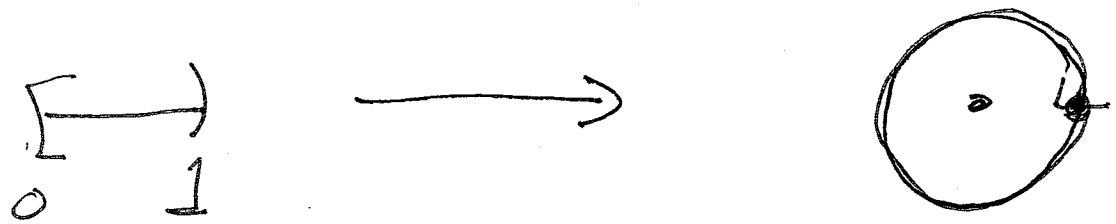
Another example.

S^1 = unit circle =

$$\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$$

with subspace topology

from $S^1 \subseteq \mathbb{R}^2_S$



$[0,1)$

$f(t) = (\cos 2\pi t, \sin 2\pi t)$

f is cont., bijective.

Is it a homeomorphism?

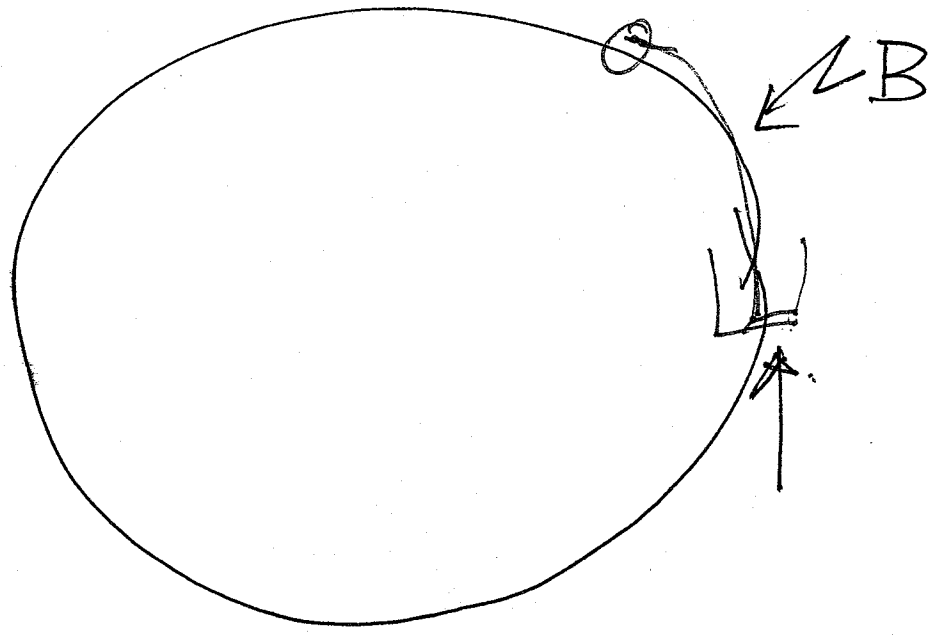
$\cancel{f(E)}$

$[0, 1/4)$ open in $[0,1)$

$$[0, 1/4) = [0,1) \cap (-1/4, 1/4)$$

$f([0, 1/4))$ is not open in S^1

So f is cont, bijective not <u>homeo</u>



To be open in Subsp Top

$\forall x \in B \ \exists U$ open in $\mathbb{R}^2$ with $x \in U$

~~$U \cap B \subseteq$~~ and

$U \cap S^1 \subseteq B$

not true for $f(0)$

$p = f(0)$