Infinite Products

$\{A_\lambda\}_{\lambda \in \Lambda}$ indexed family of topological spaces

$\prod_{\lambda \in \Lambda} A_\lambda$ is all maps $\underline{x} : \Lambda \to \bigcup_{\lambda \in \Lambda} A_\lambda$

with $x_\lambda = x(\lambda) \in A_\lambda$

Special ~~case~~ case all $A_\lambda = A$

$$\prod_{\lambda \in \Lambda} A_\lambda = A^\Lambda$$

We will often just write

$$\prod A_\lambda \text{ for } \prod_{\lambda \in \Lambda} A_\lambda$$

Most common example here.

$$\mathbb{R}^{\mathbb{Z}^+} = \text{all } \underline{x} : \mathbb{Z}^+ \to \mathbb{R}$$

$$= \text{all } (x_1, x_2, x_3, \dots)$$

$$= \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots \cdots = \mathbb{R}^\omega$$

and $[0,1]^{\mathbb{Z}_+}$ called <u>Hilbert Cube</u>

What topology do we put on $\prod A_\lambda$?

---

(1) Box topology – base is

$\mathcal{B}_B = \{ \prod_{\lambda \in \Lambda} U_\lambda : U_\lambda \text{ is open in } A_\lambda \}$

$\mathcal{B}$ is a base because $\left(\prod U_\lambda\right) \cap \prod V_\lambda = \prod (U_\lambda \cap V_\lambda)$

(2) Product topology – Sub-base

pi

$$\mathcal{S}_P = \{ \pi_\lambda^{-1}(U_\lambda) : U_\lambda \text{ open in } A_\lambda \}$$

What is the corresponding base

$$\pi_{\lambda_1}^{-1}(U_{\lambda_1}) \cap \cdots \cap \pi_{\lambda_n}^{-1}(U_{\lambda_n}) \text{ for}$$

$\{\lambda_1, \ldots, \lambda_n\} \subseteq \Lambda$ a finite subset

NOTE:

$$\pi_{\lambda_0}^{-1}(U_{\lambda_0}) = U_{\lambda_0} \times \prod_{\lambda \neq \lambda_0} A_\lambda$$

(after rearranging) so

$$\pi_{\lambda_1}^{-1}(U_{\lambda_1}) \cap \cdots \cap \pi_{\lambda_n}^{-1}(U_{\lambda_n})$$

$$= U_{\lambda_1} \times U_{\lambda_2} \times \cdots \times U_{\lambda_n} \times \prod_{\substack{\lambda \neq \lambda_i \\ i=1,\dots,n}} A_\lambda$$

So A base element in the product topology is

$$\prod_{\lambda \in \Lambda} U_\lambda$$ where all but finitely many U_λ are A_λ

A base element in the box topology is $$\prod_{\lambda \in \Lambda} U_\lambda$$ with no restrictions

---

Box $\supseteq$ product, Box is finer than product.

In $\mathbb{R}^{\mathbb{Z}_+}$

BASE for Box:

$$(a_1, b_1) \times (a_2, b_2) \times \cdots\cdots$$

BASE for product:

$$(a_1, b_1) \times \cdots \times (a_n, b_n) \times \prod_{i=n+1}^{\infty} \mathbb{R}$$

Facts (1) subspaces work as expected

In both topolgies (2) product of HD is HD

(3) $A_\lambda \subseteq X_\lambda$ work in $\prod X_\lambda$

Then $\prod \overline{A_\lambda} = \overline{\prod A_\lambda}$

Proof of (3).: $\cancel{\underline{x}}$ $(\subseteq)$ $\underline{x} \in \prod \overline{A_\lambda}$

and $\underline{x} \in U = \prod U_\lambda$ (open in $\prod X_\lambda$)

then $x_\lambda \in U_\lambda$ since $x_\lambda \in \overline{A_\lambda}$ then

$\exists\ y_\lambda \in U_\lambda \cap A_\lambda$. Do for every λ)

$\underline{y} \in U \cap \prod A_\lambda \Rightarrow \underline{x} \in \overline{\prod A_\lambda}$

($\supseteq$) $\underline{x} \in \overline{\prod A_\lambda}$ and V_{λ_0} open in X_{λ_0} and $x_{\lambda_0} \in V_{\lambda_0}$. Now

$\pi_{\lambda_0}^{-1}(V_{\lambda_0})$ open in $\prod X_\lambda$

so, it contains a point

$\underline{y} \in \prod(A_\lambda) \Rightarrow$

$y_{\lambda_0} \in V_{\lambda_0} \cap A_{\lambda_0} \Rightarrow$

$x_{\lambda_0} \in \overline{A_{\lambda_0}}$, λ_0 was arbitrary

$\underline{x} \in \prod \overline{A_\lambda}$. ∎

DEF: $f : A \to \prod_{\lambda \in \Lambda} X_\lambda$ has

components $f_\lambda : A \to X_\lambda$

with $f_\lambda(a) = (f(a))_\lambda$

OR $\pi_{\lambda_0} : \prod X_\lambda \to X_{\lambda_0}$

via $\pi_{\lambda_0}(\underline{x}) = x_{\lambda_0}$

then $f_\lambda = \pi_\lambda \circ f$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "f_{\lambda_0}"'] & \prod X_\lambda \arrow[d, "\pi_{\lambda_0}"] \\
& X_{\lambda_0}
\end{tikzcd}$$

Theorem $f : A \to \prod X_\lambda$

(a) ~~If~~ Assume $\prod X_\lambda$ has the product top.

f is cont $\Leftrightarrow$ each f_λ is cont

(b) not neccessarily true when

$\prod X_\lambda$ has the box topology.

(a) ($\Rightarrow$) The projection $\pi_{\lambda_0} : \prod X_\lambda \to X_{\lambda_0}$

is continuous since $\pi_{\lambda_0}^{-1}(U)$ is

open in $\prod X_\lambda$ since it is a base

element when U is open in X_{λ_0}.

and $f_{\lambda_0} = \pi_{\lambda_0} \circ f$, composition of cont. funct

($\Leftarrow$) Suffice's to check

$f^{-1}(B)$ is open in A for every

sub-base element in product topolog

or $B = \pi_\lambda^{-1}(U_\lambda)$ with U_λ open in X_λ

$f^{-1}(B) = f^{-1}\pi_\lambda^{-1}(U_\lambda) = f_\lambda^{-1}(U_\lambda)$ is

since $f_\lambda = \pi_\lambda \circ f$

open in A by assumption that

f_λ is continuous. $\blacksquare$

counter example with the box topology.

$$f : \mathbb{R} \to \mathbb{R}^{\mathbb{Z}_+}$$

as $f(t) = (t, t, t, t, \ldots)$

So $f_n(t) = t$ ~~$\forall t \in \mathbb{R}$~~ $\forall n \in \mathbb{Z}^+$

$$U = \prod_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = (-1, 1) \times (-1/2, 1/2) \times \cdots$$

is open in the box topology.

If f was cont, then $f^{-1}(U)$ open in $\mathbb{R}$ and contains zero. $\exists \delta > 0$ with

$(-\delta, \delta) \subseteq f^{-1}(U)$.

or $f((-\delta, \delta)) \subseteq U$

$$f((-\delta,\delta)) \subseteq U$$

$$(-\delta,\delta) = \pi_n f((-\delta,\delta)) \subseteq \pi_n(U) = \left(-\tfrac{1}{n}, \tfrac{1}{n}\right)$$

$$f(t) = (t, t, t, t, \ldots)$$

$$f((-\delta,\delta)) = (-\delta,\delta) \times (-\delta,\delta) \times \ldots$$

~~True~~ can't be true for all $n \in \mathbb{Z}^+$ since $\delta > 0$