Given a metric it yields a base

\[ \exists B_x(r): x \in X, r > 0 \exists \]

Last time when two metrics yield the same topology.

**Discrete metric**

\[
d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}
\]

⇒ the discrete topology every pt is open
Example for \( \mathbb{R}^{2+} \)

What metric should we try to generalize from \( \mathbb{R}^n \)?

\[ y, x \in \mathbb{R}^{2+} \]

\[ d(x, y) = \left( \sum_{i=0}^{\infty} (x_i - y_i)^2 \right)^{1/2} \]

Could be infinite

\[ \sum \text{ works on } \ell^2(\mathbb{R}) = \{ x \in \mathbb{R}^{2+} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \} \]

or \( d(x, y) = \sup_{i \geq 1} |x_i - y_i| \), \( i \in \mathbb{Z}^{+} \)

also could be infinite
Use the bounded metric on $\mathbb{R}$.

\[ I(x, y) = \begin{cases} \sum d(x, y) & \text{when } d(x, y) \leq 1 \\ \geq 1 & \text{when } d(x, y) \geq 1 \end{cases} \]

generates the same topology as $d$

\[ d(x, y) = |x - y| \]

Two metrics on $\mathbb{R}^2^+$

\[ \rho(x, y) = \sup \sum \overline{d}(x_i, y_{i+1}), \quad i \in \mathbb{Z}^+ \]

\[ D(x, y) = \sup \sum \frac{\overline{d}(x_i, y_{i+1})}{2^i}, \quad i \in \mathbb{Z}^+ \]
DEF: The topology induced by $\rho$ is called the uniform topology.

Theorem:
(a) $\rho$ and $D$ are metrics
(b) $D$ induces the product topology on $\mathbb{R}^2$
(c) box $\neq$ uniform $\neq$ product.

R.K.: So $\rho$ and $D$ induce different topologies on $\mathbb{R}^2$.
Proof (b)

Two steps:

(1) Product is D-topology.

(2) D-topology is product in product colimit.

Claim $x \in U$ is obvious.

$V = \{ x_1 \leq x_1 + \epsilon, x_2 \leq x_2 + \epsilon, \ldots, x_n \leq x_n + \epsilon \}$
In general,
\[ D(x, y) \leq \max \left( \frac{\overline{d}(x_i, y_i)}{4}, \frac{\overline{d}(x_n, y_n)}{n} \right) \]

Since for the first \( N \)-terms by def

for large \( n > N \), \( p \leq 1 \) \( \forall i \)

so \( \overline{d}(x_n, y_n) \leq \frac{1}{n} < \frac{1}{n} \)

Now if \( y \in V \Rightarrow D(x, y) < 3 \)

since \( \frac{\overline{d}(x_n, y_n)}{n} < \frac{3}{n} \) for \( n = \frac{1}{n}, n \)

and \( \frac{1}{n} < 3 \).
2) $D_{top} = \text{product.}$

Let $U$ be a base element in product topology or

$U = \prod_{i \in \mathbb{Z}} U_i$ when $U_i \neq \emptyset$

only for $i = d_1, \ldots, d_n$. The task

is given $x \in U$ find $B_{\epsilon}(x) \subseteq U$

($B_{\epsilon}$ in $D$-metric).

We can find $\varepsilon_i$ with $\varepsilon_i \leq \frac{1}{2}$

$(x_{\Delta+\varepsilon}, x_{\Delta+\varepsilon}) \subseteq U_i$ if $i \neq d_1, \ldots, d_n$.
and \( \exists n \in \mathbb{N} \) so \( x < 1 \) ?

\[ x - y \leq \frac{7}{3} \]

So \( x \geq \frac{7}{3} \) and \( z > \frac{7}{3} \).

\[ \frac{7}{3} \geq z \]

However, for \( l = \frac{7}{3} \), \( \frac{7}{3} = \frac{7}{3} \).

So by the condition \( 3 > \frac{7}{3} \).

So let \( x \geq \frac{7}{3} \) min \( n \).
Proper containment is $HW$.

Unit is product. Pick $\Pi_U$ base.

Element in product topology so

$U_i$ open in $\mathbb{R}$ and $U_i \neq \mathbb{R}$ only.

For $x, y \in \mathbb{R}_n$. For each $i = x_i, y_i$.

Choose $\varepsilon$ so that $(x_i - \varepsilon, x_i + \varepsilon) \subseteq U_i$.

and let $\varepsilon = \min\{\varepsilon, \varepsilon, \varepsilon\}.$
I.e. \( \exists p(x, z) \neq 0 \)

\[ \implies \exists (x, z) : \forall x. \]

So \( z \in \pi \cup \Lambda \)

So \( B^\theta(x) \subseteq \pi \cup \Lambda \)

ball in \( \rho \)-metric
box is unit

Given $B^p_\frac{1}{2} (x)$ then

$$\prod_{l=1}^n (x_l - \frac{1}{2}\varepsilon, x_l + \frac{1}{2}\varepsilon)$$
is a box inside $B^p_\frac{1}{2} (x)$. 

Many nice properties that are just true for metric spaces.

**Continuity**

**Theorem:**

If $f: (X, d_X) \to (Y, d_Y)$ is continuous so that $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in X$, then $f$ is uniformly continuous on $X$.

For any $\varepsilon > 0$, let $\delta = \varepsilon$. Then for all $x, y \in X$ with $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) \leq \varepsilon$, so $f$ is uniformly continuous on $X$.

Proud to be here. One way.
Two other results proved next time.

1) \( f: (X,d_X) \rightarrow (Y,d_Y) \) is continuous

\[ \iff \left( x_n \rightarrow x \text{ in } X \implies f(x_n) \rightarrow f(x) \text{ in } Y \right) \]

2) \( (X,d) \) metric, \( A \subseteq X \)

\( x \in \overline{A} \iff \exists \; \exists n \in \mathbb{N} \quad x_n \in A \quad x_n \rightarrow x \)