Given a metric it yields a base

$$\{ B_\varepsilon(x) : x \in X, \varepsilon > 0 \}$$

Last time when two metrics yield the same topology.

Discrete metric

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

$\Rightarrow$ the discrete topology every pt is open

Example for $\mathbb{R}^{\mathbb{Z}_+}$

What metric should we try to generalize from $\mathbb{R}^n$.

$\underline{y}, \underline{x} \in \mathbb{R}^{\mathbb{Z}_+}$

$d(\underline{x}, \underline{y}) = \left(\sum_{i=0}^{\infty} (x_i - y_i)^2 \right)^{1/2}$, could be infinite

[works on $\ell^2(\mathbb{R}) = \{ \underline{x} \in \mathbb{R}^{\mathbb{Z}^+} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \}$]

x_i

---

or $d(\underline{x}, \underline{y}) = \sup \{ |x_i - y_i| : i \in \mathbb{Z}^+ \}$

also could be infinite

Use the bounded metric on $\mathbb{R}$.

$$\bar{d}(x,y) = \begin{cases} d(x,y) & \text{when } d(x,y) \le 1 \\ 1 & \text{when } d(x,y) \ge 1 \end{cases}$$

generates the same topology as d

$(d(x,y) = |x-y|)$

Two metrics on $\mathbb{R}^{\mathbb{Z}_+}$

$$\rho(\underline{x},\underline{y}) = \sup\{\bar{d}(x_i,y_i) : i \in \mathbb{Z}_+\}$$

$$D(\underline{x},\underline{y}) = \sup\left\{\frac{\bar{d}(x_i,y_i)}{i} : i \in \mathbb{Z}_+\right\}$$

DEF: The topology induced by ρ is called the uniform topology.

Theorem (a) ρ and D are metrics

(b) D induces the product topolog on $\mathbb{R}^{\mathbb{Z}_+}$

(c) box $\supsetneq$ uniform $\supsetneq$ product.

RK: So ρ and D induce different topologies on $\mathbb{R}^{\mathbb{Z}_+}$

Proof (a) is easy.

(b) Two steps: (1) product $\supseteq$ D-topology

(2) D-topology $\supseteq$ product

(1) Give $\mathcal{U}$ open in (X, D) and $x \in \mathcal{U}$, $\exists$ V open in product with $x \in V \subseteq \mathcal{U}$. Find $B_\varepsilon(x) \subseteq \mathcal{U}$

Pick $\frac{1}{N} < \varepsilon$ and then let

$$V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N - \varepsilon) \times \mathbb{R} \times \mathbb{R}$$

Claim $V \subseteq B_\varepsilon(x) \subseteq \mathcal{U}$.

$x \in V$ is obvious

In general,

$$\rightarrow D(x,y) \leq \max\left(\frac{\bar{d}(x_1,y_1)}{1}, \dots, \frac{\bar{d}(x_N,y_N)}{N}, \frac{1}{N} \right)$$

since for the first N-terms by def

for large $n > N$, $\bar{d} \leq 1 \quad \forall i$

so $\dfrac{\bar{d}(x_n,y_n)}{n} \leq \dfrac{1}{n} < \dfrac{1}{N}$

Now if $y \in V \Rightarrow D(x,y) < \varepsilon$

since $\dfrac{\bar{d}(x_n,y_n)}{n} < \dfrac{\varepsilon}{n}$ for $n = 1, \dots, N$

and $\dfrac{1}{N} < \varepsilon$.

(2) D- top $\supseteq$ product.

U be a base element in product totop

or $U = \prod_{i \in \mathbb{Z}_+} U_i$ when $U_i \neq \mathbb{R}$

only for $i = \alpha_1, \dots, \alpha_n$. Retask

is given $x \in U$ find $B_\varepsilon(x) \subseteq U$

(B_ε in D-metric).

We can find ε_i with $\varepsilon_i \leq 1$

$(x_i - \varepsilon_i, x_i + \varepsilon_i) \subseteq U_i \quad i = \alpha_1, \dots, \alpha_n$

Let $\varepsilon = \min\left\{\frac{\varepsilon_i}{i} ; i = \alpha_1, \dots, \alpha_n\right\}$

So by def if $y \in B_\varepsilon(x)$

$$\Rightarrow \frac{\bar{d}(x_i, y_i)}{i} \le D(x, y) < \varepsilon.$$

However, for $i = \alpha_1, \dots, \alpha_n$, $\varepsilon \le \frac{\varepsilon_i}{i}$

So $\bar{d}(x_i, y_i) < \varepsilon_i \le 1$ so.

$|x_i - y_i| < \varepsilon_i$ So $y \in \prod U_i$

and $B_\varepsilon(x) \subseteq \prod U_i$ $\blacksquare$

(c) box $\supsetneq$ unif $\supsetneq$ product
$\qquad\qquad\quad\;\; \| \qquad\qquad\quad \|$
$\qquad\qquad$ ρ-top $\qquad$ D-top

Proper containment is HW.

unif $\supseteq$ product $\quad$ pick $\prod U_i$ base element in product topology so U_i open in $\mathbb{R}$ and $U_i \neq \mathbb{R}$ only for $\alpha_1, \ldots, \alpha_n$. For each $i = \alpha_1, \ldots, \alpha_n$ chose ε so that $(x_{\alpha_i} - \varepsilon_i, x_{\alpha_i} + \varepsilon_i) \subseteq U_i$ and let $\varepsilon = \min(\{\varepsilon_i\})$.

If $\rho(x,z) < \varepsilon$

$\Rightarrow \bar{d}(x_\alpha, z_\alpha) < \varepsilon \quad \forall \alpha.$

so $z \in \prod U_\alpha$

so $B_\varepsilon^{\rho}(x) \subseteq \prod U_\alpha$

ball in ρ-metric

box $\supseteq$ unif

Given $B^{\rho}_{\varepsilon}(x)$ then

$$\prod_{i=1}^{\infty}\left(x_i - \tfrac{1}{2}\varepsilon,\ x_i + \tfrac{1}{2}\varepsilon\right)$$ is a box

nbd inside $B^{\rho}_{\varepsilon}(x)$ ∎.

Many nice properties that are just true for metric spaces.

$\boxed{\text{Continuity}}$ **Theorem:**

$f : (X, d_X) \to (Y, d_Y)$ is continuous

$\Leftrightarrow \Big(\forall \varepsilon > 0 \ \exists \delta > 0$ so that
$d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon \Big)$

~~$\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ so that~~

proved before one way

Two other results proved next time.

① $f : (X, d_X) \to (Y, d_Y)$ is continuity

$\iff$ ($x_n \to x$ in X implies $f(x_n) \to f(x)$ in Y)

② (X, d) metric, $A \subseteq X$

$x \in \bar{A} \iff \exists \{x_n\} \subseteq A$ with $x_n \to x$