Quotient Topology — no longer just metric space

Top 18b

Given X with an equivalence $\sim$

Let $[x] = \{y \in X : y \sim x\}$ is equivalence class.

Let $X^* = X/\sim = \{[x] : x \in X\}$

set of equivalence classes.

Define $\pi : X \to X/\sim$ as

$\cancel{\pi(x) \to \{X\}}$ $\pi(x) = [x]$

pt $\to$ equiv class.

The topology on $X/\sim$ in called the quotient topology and $X/\sim$ is called the identification or decomposition space..

<u>DEF</u>: U is open in $X/\sim$ $\Longleftrightarrow$ $\pi^{-1}(U)$ is open in X.

<u>OR</u> The quotient topology is the largest topology that makes π continuous.

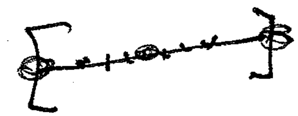
Examples

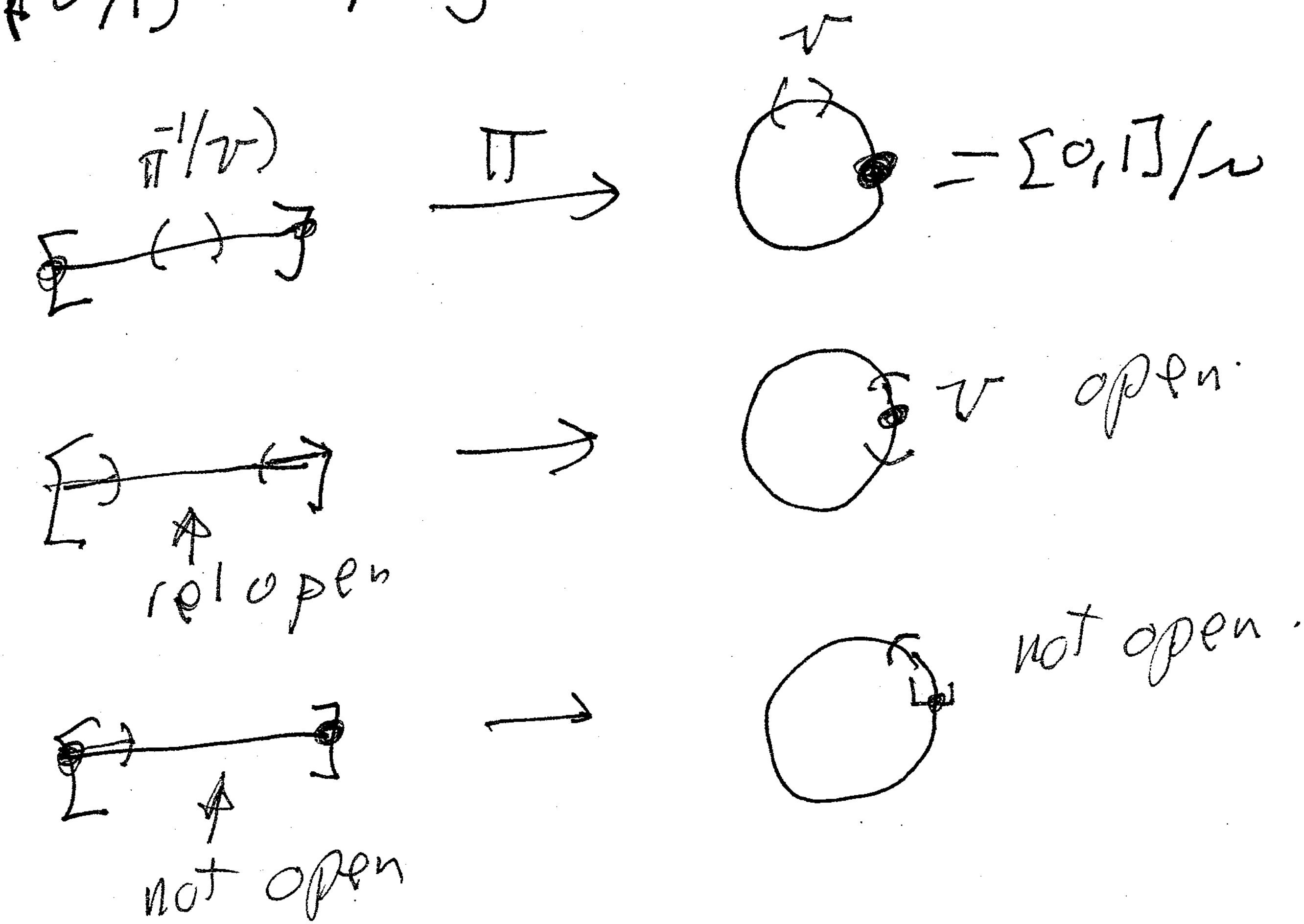
$X = [0,1]$

$0 \sim 1$

$x, y \in (0,1)$ $\quad x \sim y \iff x = y$

$X/\sim$ is homeomorphic to ~~S^1 with~~ $S^1 \subseteq \mathbb{R}^2$ with the subspace topology

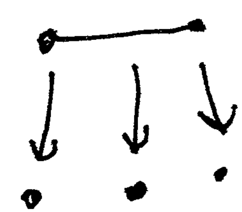




<u>CAUTION</u>: Quotient topology of even nice spaces is sometimes not Hausdorff.

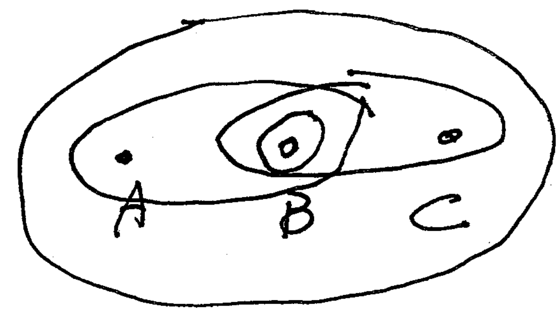
<u>Example 1</u>: $X = [0,1] \subseteq \mathbb{R}$ with subspace topology.
Let $\sim$ have three equivalence classes

$A = \{0\}$ $C = \{1\}$
$B = (0,1)$



So $X/\sim\ = \{A, B, C\}$

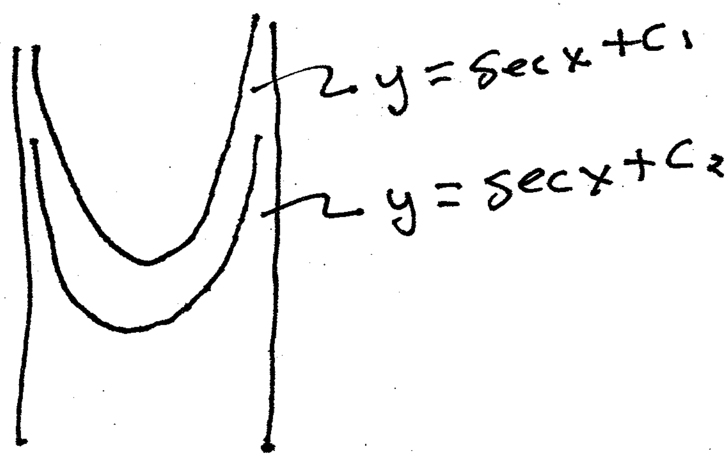
$\pi^{-1}(A) = \{0\}$ not open
$\pi^{-1}(B) = (0,1)$ open
$\pi^{-1}(C) = \{1\}$ not open
$\pi^{-1}(A \cup B) = [0,1)$ open
$\pi^{-1}(B \cup C) = (0,1]$ open
$\pi^{-1}(A \cup C) = \{0\} \cup \{1\}$ not open



From 1st intro to Topologies.

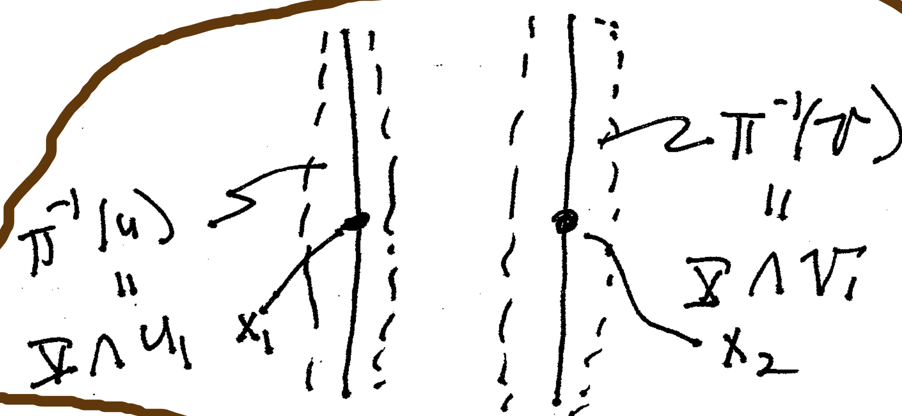
Example 2 $X = [-\pi/2, \pi/2] \times \mathbb{R} \subseteq \mathbb{R}^2$ with subspace topology.

Let $\sim$ have equivalence classes $\{-\pi/2\} \times \mathbb{R}$, $\{\pi/2\} \times \mathbb{R}$, graph of $y = \sec x + C$ for each $c \in \mathbb{R}$



Claim, in $X/\sim$ there do not exist disjoint open sets separation $p = \pi(\{-\frac{\pi}{2}\} \times \mathbb{R})$ and $q = \pi(\{+\frac{\pi}{2}\} \times \mathbb{R})$

Argument: Say they are separated $p \in U$ and $q \in V \Rightarrow \pi^{-1}(U)$ is an open set containing $\{-\frac{\pi}{2}\} \times \mathbb{R}$ and $\pi^{-1}(V)$ is an open set containing $\{\frac{\pi}{2}\} \times \mathbb{R}$



U_1 and V_1 open in $\mathbb{R}^2$

Thus, if we pick two points say $x_1 = (-\pi/2, 0)$ $x_2 = (\frac{\pi}{2}, 0)$

There are balls $B_{\varepsilon_1}(x_1) \subseteq \pi^{-1}(U)$ and $B_{\varepsilon_2}(x_2) \subseteq \pi^{-1}(V)$

But there is a C so that the graph of $\sec x + C$ intersects both balls and so $\pi(\sec x + C) \in U \cap V$ a contradiction.