Connectness

$(X, T)$ a separation of $X$ is a pair of disjoint, nonempty, open sets $U$ and $V$ with $U \cup V = X$. $X$ is connected if it has no separation.

Alternatively $(X, T)$ is connected if its only subsets which are open and closed are $X$ and $\emptyset$. 
For subspaces there is a closure condition.

**DEF:** \( y \subseteq X \) a separation of \( Y \) is a pair of relatively open sets \( U \) and \( V \) with \( U \cap V = \emptyset \) \( U \cup V = Y \).

**Lemma:** \( A \) and \( B \) give a separation \( \emptyset \neq y \subseteq X \iff A \cup B = X, A \cap B = \emptyset \) and \( A \) and \( B \) are nonempty.

\( \text{Cl}_X(A) \cap B = \emptyset \) \( \text{Cl}_X(B) \cap A = \emptyset \)

or neither contains a limit point of the other.
\((\Rightarrow)\) Assume \(A\) and \(B\) form a separation of \(y \in X\). \(\Rightarrow\) \(A\) is both open and closed in \(Y\). We proved that:

Then \(\text{Cl}_Y(A) = \text{Cl}_X(A) \cap Y\). But \(A\) is closed in \(Y\) \(\Rightarrow A = \text{Cl}_X(A) \cap Y\).

but \(A \cap B = \emptyset\), \(A \cup B = Y\) \(\Rightarrow \text{Cl}_X(A) \cap B = \emptyset\)

but \(\text{Cl}_X(A) = A' \cup A\) and \(A \cap B = \emptyset\)

\(\Rightarrow A' \cap B = \emptyset\). Switch roles of \(A\) and \(B\).
Say \( A \cup B = Y \) and \( A \cap B = \emptyset \).

\[
\begin{align*}
\emptyset \subseteq (B) \land A &= \emptyset \\
\emptyset \subseteq (B) \land y &= B \\
\emptyset \subseteq (A) \land (B) &= A \\
A &= B \cup (B) \\
x \in (A) \land (B) &= A \\
A \cap B &= \emptyset \\
A \cup B &= Y \\
A \text{ and } B \text{ are preclosed}
\end{align*}
\]

Thus \( A \) and \( B \) are open.
Examples:

1) \([-1, 0) \cup (0, 1] = Y \subseteq \mathbb{R}.

\[\overline{A} = \mathbb{R} \setminus (-1, 0) \ \land \ B = \emptyset \implies \text{separation}
\]
\[\overline{B} = (0, 1] \setminus A = \emptyset \implies \text{not connected}.
\]

\[\text{But } \overline{A} \cap \overline{B} \neq \emptyset, \text{ but that's OK.}
\]

2) \([-1, 1] \text{ connected (proved later)}

3) \[Y = \mathbb{Q} \cap [-1, 1] \text{ not connected.}
\]
\[A = [0, \sqrt{2}] \quad B = (\frac{\sqrt{2}}{2}, 1]
\]

\[\text{separation = disconnection}\]
\( \mathbb{R} \times \mathbb{R}^2 \cup \Sigma(x, \sqrt{x}) : x > 0 \)

Not connected
\[ B = A \cup B \]
\[ A = x \]

Topologist's sine curve

Path connected

Not connected

\[ \exists 0 < \varepsilon \leq 4 \]

\[ \exists 0 \leq x < 4 \]

\[ 0. \exists x, \sin \frac{1}{x} \]

\[ (0. \exists 0.5 \times 3 \leq 1 \]
Lots of basic properties of connectedness

Lemma: If $C$ and $D$ form a separation of $X$ and $y \in X$

is connected $\implies$ either $y \in C$

or $y \in D$.

Proof: If not, $(C \cup y)$ and $(D \cup y)$

form a separation of $X$. 
Lemmas: $\exists x \exists z \forall x \in A$, each $A_x$

Connected and $\bigwedge_{x \in I} A_x \neq \emptyset$

$\Rightarrow \bigvee_{x \in I} A_x$ is connected.

Proof: Assume $B$ and $C$ form a sep. $y = \bigvee_{x \in I} A_x$. If $p \in \bigwedge_{x \in I} A_x$

$\Rightarrow p \in C$ or $p \in D$, assume $p \in C$. Now $A_x$ is connected, so

by previous lemma $A_x \subset C$ or $A_x \subset D$.
Since \( p \in C \), \( A_\lambda \subseteq C \) true for all \( \forall \lambda \) so \( y = \bigcap \bigcup \bigcap A_\lambda \subseteq C \)

\[ \Rightarrow D \neq \emptyset \text{ contradiction.} \]

Example \( \exists \) lines \( y = mx \): \( m \in \mathbb{Q} \)

\[ \Rightarrow \text{each is } L_m \]

\( o \in \bigcap L_m \), \( L_m \text{ conn. } \)

\[ m \notin \mathbb{Q} \]

\[ \Rightarrow U L_m \text{ is connected} \]

Example
Lemma: $A$ is connected and $A \subseteq B \Rightarrow B$ is also connected.

Corr: $A$ connected $\Rightarrow A$ connected.

Example:

\[ \text{Connected set doesn't disconnect.} \]