$\boxed{\text{Connectedness}}$

L is a linear continuum if

(1) $|L| > 1$
(2) strict linear order
(3) least upper bound property
(4) (archimedian property).
$x < y \Rightarrow \exists z \;\; x < z < y$

(in $\mathbb{R}$, no smallest number)

example: $(-\infty, 0] \cup [1, \infty]$
satisfies (1), (2), (3) not 4
and is connected.

Th'm Put order topology on L. a linear cont $\Rightarrow$ intervals and rays are connected.

Proof (sketch) Define $Y \subseteq L$ as <u>convex</u> if $a, b \in Y \Rightarrow [a,b] \subseteq Y$. Actually prove that convex subsets of L are conn. We reduce to the case of an interval.

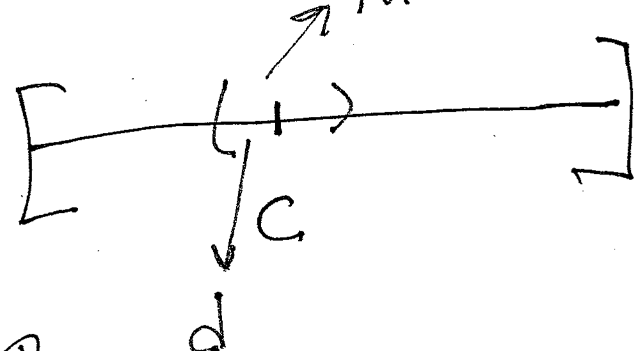
So assume $Y \subseteq L$ is convex and has a separation $A \cup B = Y$. Pick $a \in A$ and $b \in B$ so $[a,b] \subseteq Y$ by the convexity assumption.

Then $A_0 = A \cap [a,b]$ and $B_0 = B \cap [a,b]$

is a separation of $[a,b]$.

Let $c = \sup(A_0)$, where is c?

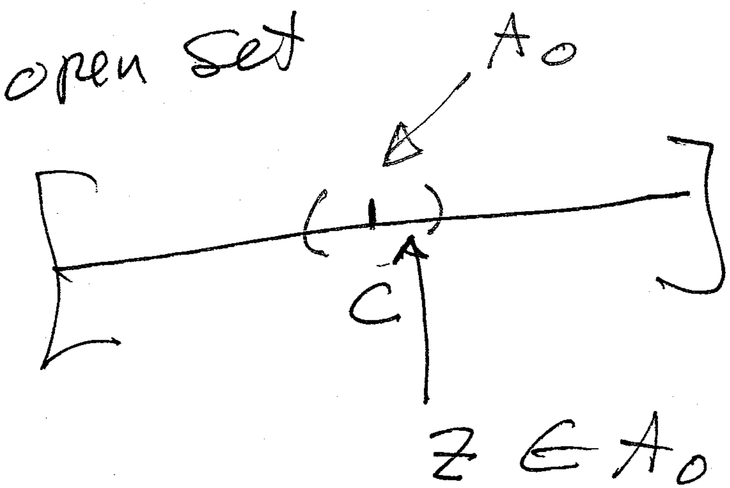
$c \in B_0$ and open set



$d < c$ and
$d \in B_0$ so d is a smaller upper
bound for A_0

(note $c = b$ is possible)

$c \in A_0$, an open set



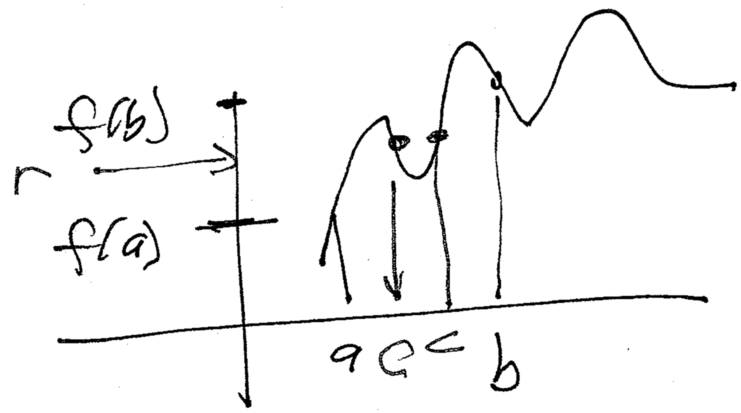
$\Rightarrow z > c$ which was an upper bound.

($c = a$ also possible).

c (w, v)

$w < c < v$
$\uparrow$
z

Calculus intermediate value theorem.

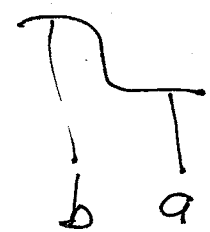


$f: \mathbb{R} \to \mathbb{R}$

continuous

If $f(a) < f(b)$ and

$f(a) < r < f(b) \implies$



c between a and b. with

$f(c) = r$.

$\boxed{\text{Generalized IVT.}}$

$f: X \to Y$ continuous

X is connected.

Y has a strict linear order.

Assume $f(a) < f(b)$ $a, b \in X$

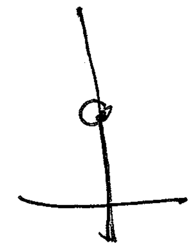
and r is such that $f(a) < r < f(b)$

$\Rightarrow \exists\, c \in X$ with $f(c) = r$.

Suppose no such c exists.

$A = f(X) \cap (-\infty, r)$

$B = f(X) \cap (r, \infty)$



$A \cup B = f(X)$ since no c exists with $f(c) = r$. and A and B are open in $f(X)$., $A \cap B = \emptyset$

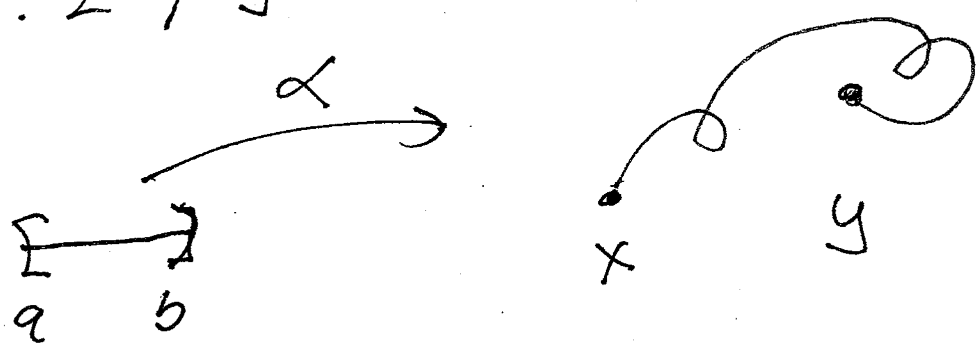
$\Rightarrow$ A and B form a separation of $f(X)$ which is connected since f is cont, X is conn.

contradiction. $\blacksquare$

Path connectedness

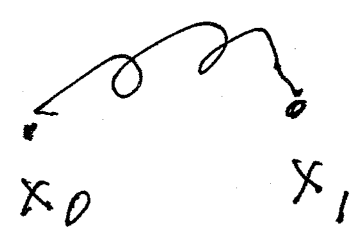
Given $x, y \in X$ a __path__ from x to y is a continuous function

$$\alpha : [a,b] \to X \quad \text{with } \alpha(a) = x \quad \alpha(b) = y$$



Alternatively, $\alpha : [0,1] \to X$ with

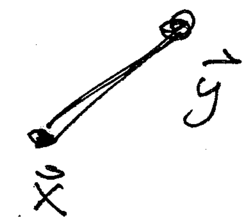
$$\alpha(i) = x_i \quad i = 0, 1$$



X is <u>path connected</u> if $\forall x, y \in X$ $\exists$ path x to y.

<u>examples</u>

(1) $\mathbb{R}^n$ $\quad \vec{x}, \vec{y} \in \mathbb{R}^n$

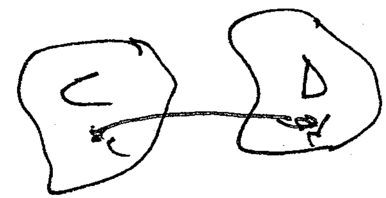


let $\alpha(t) = (1-t)\vec{x} + t\vec{y}$

<u>Lemma</u> Path conn $\Longrightarrow$ conn.

__Proof__: Let $X = C \cup D$ be a separation of X which is path conn.

Pick $c \in C$ $d \in D$ and α a path c to d.



But $\alpha([0,1])$ is connected as the cont image of a conn set

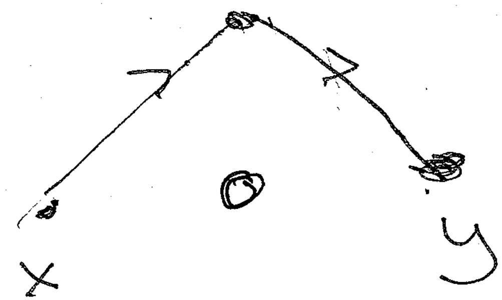
$\Rightarrow$ $\alpha([0,1]) \subseteq C$ or $\alpha([0,1]) \subseteq D$ in either case it can't connect c to d. $\blacksquare$

Examples

(2) $\mathbb{D}^n = \{\vec{x} \in \mathbb{R}^n : \|x\| \leq 1\}$

$\alpha(t) = \cancel{\alpha}\,(1-t)\vec{x} + t\vec{y}$

(3) $\mathbb{R}^n - \{0\}$ $\quad n > 1$



(4) $\mathbb{R}^3 - \{x\text{-axis}\}$

(5) $S^n \subseteq \mathbb{R}^{n+1}$, $S^n = \{\vec{x} : \|x\| = 1\}$

$n \geq 0$

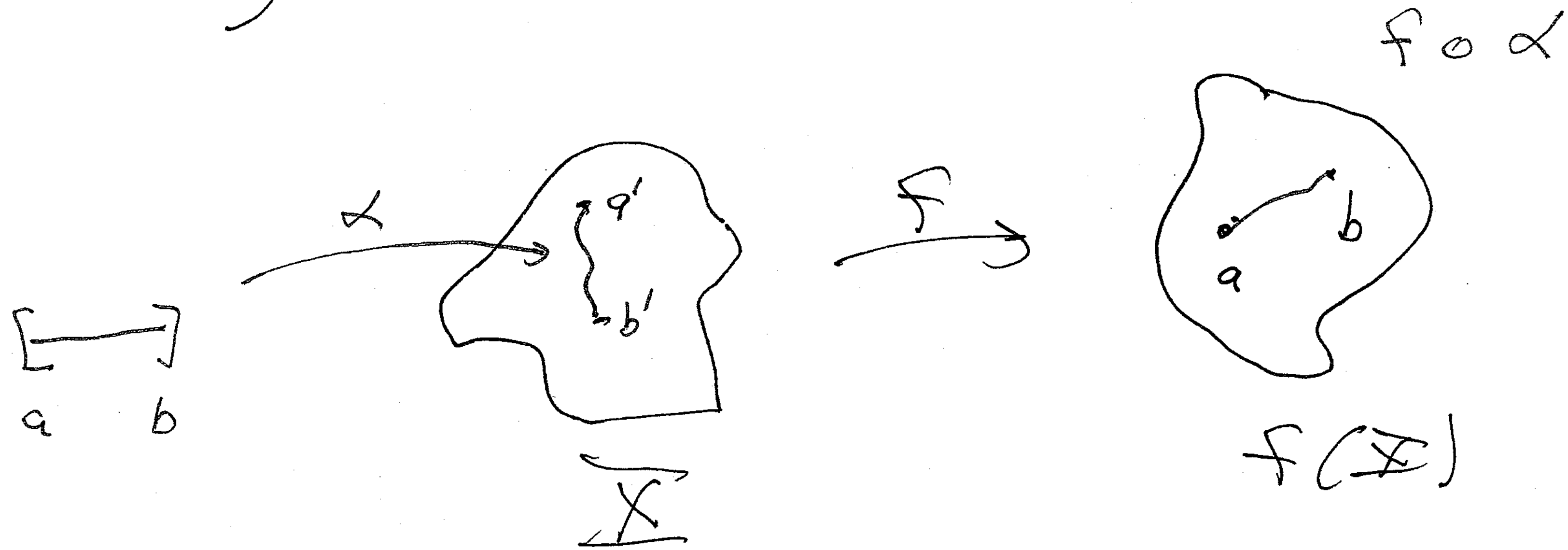
define $g : \mathbb{R}^{n+1} - \{0\} \to S^n$

$g(\vec{x}) = \dfrac{\vec{x}}{\|\vec{x}\|}$ is cont.

$\Rightarrow$ S^n is conn.

__Th'm__: If X is path connected, $f: X \to Y$ is ~~connect~~ continuous

$\Rightarrow f(X)$ is path connected.



$f \circ \alpha$ is a path.

Sometimes α is the path. Somes $\alpha([0,1])$ is the path.