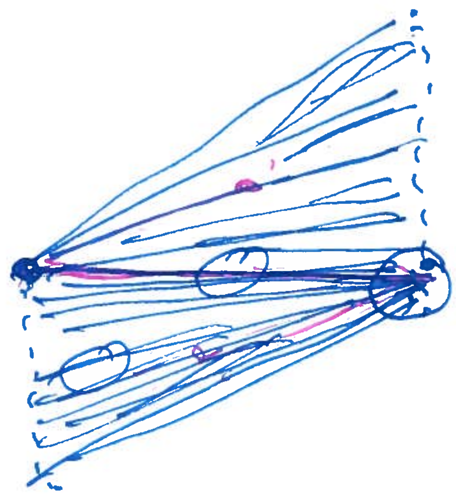
X is locally path connected if

$\forall$ open nhd U @ x $\exists$ open, path connected V

$x \in V \subseteq U$

Can X be not loc conn. at every point but still be connected?

Attractors | Strange



connect $(0,0)$
to $\{1\} \times (\mathbb{Q} \cap [0,1])$

connect $(1,0)$
to $\{0\} \times (\mathbb{Q} \cap [-1,0])$

path connected
nowhere loc conn.

MAIN RESULT

- If X is locally path connected
  $\Rightarrow$ its components and path components are the same.

<u>CORR:</u> X is loc path conn and conn.
$\Rightarrow$ path conn.

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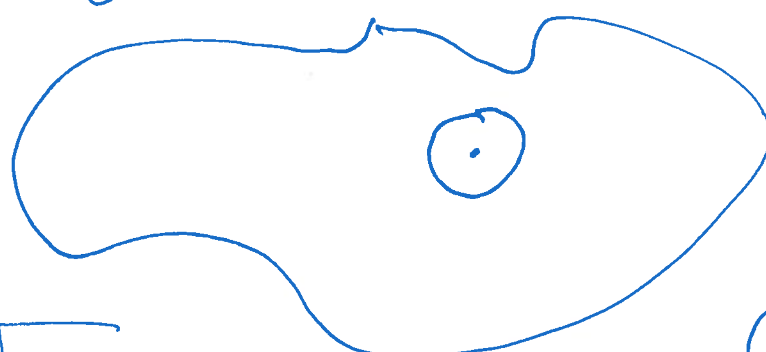
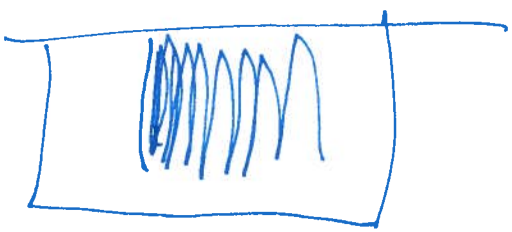
eg// open sets in $\mathbb{R}^n$ are locally path conn

at ~~x~~ x

$\exists\ B_\varepsilon(x) \subseteq V$

$B_\varepsilon(x)$ is path conn.





<u>**Lemma**</u> X is loc path conn $\iff$ $\forall$ open U each path comp of U is open in X.

Proof

($\Rightarrow$) U open, C a component of U.

$x \in C$, by loc con $\exists$ open conn

V with $\cancel{x \in V \subseteq C}$ $x \in V$.

But V is connected, $x \in C \Rightarrow V \subseteq C$

Thus each $x \in C$ has an open V

with $x \in V \subseteq C$ $\Rightarrow$ C is open.

($\Leftarrow$) Assume comp of open sets are open. $x \in X$, U an nbd of x

Let C be comp of U containing x

$\Rightarrow$ C is open, conn and $x \in C$

So X is loc connected. $\blacksquare$

CORR Let $U = X$ is open so path comp and comp of X are open when X is loc conn.

__Theorem__. $(X, \mathcal{T})$.

(1) each component is the disjoint union of path components.

(2). If X is locally path con $\Rightarrow$ comp and path comp are the same.

__Proof__ (1) $x \simeq y \Rightarrow x \sim y$

so $\sim$ equiv classes are the disjoint union of $\simeq$ classes.

Pick P a path component inside C a component

we want $P = C$. Assume not,

$P \subsetneq C$.

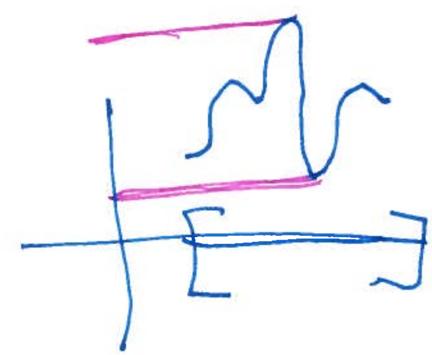
$$\Rightarrow C = P \amalg Q$$

$Q = \bigcup P_x$, P_x path comp of C different that P.

But by the lemma, Q is open in X and P is open so P, Q gives a ~~sep~~ separation of the connected C. contradiction. $\blacksquare$

$\boxed{\text{Compactness}}$

Extreme value theorem. of Calculus.



$f : [0,1] \to \mathbb{R}$

continuous.

$\exists\ x_{min}, x_{max} \in [0,1]$ with.

$f(x_{min}) \leq f(x) \quad \forall x \in [0,1]$

$f(x_{MAX}) \geq f(x) \quad$ "

"achieves its MAX and min"

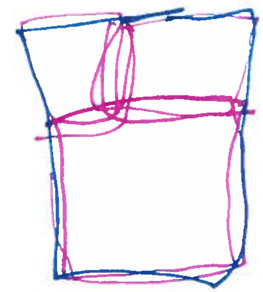
needs domain closed and bounded in $\mathbb{R}^n$.

This is equivalent to compactness in $\mathbb{R}^n$.

DEF:

(1) $A = \{A_\lambda\}_{\lambda \in \Lambda}$ such that $X = \bigcup_{\lambda \in \Lambda} A_\lambda$ is called a <u>cover of X</u>

- $\mathcal{A}$ is an open cover if each A_λ is open.



- X is <u>compact</u> if every open cover has a finite subcover i.e. a finite subcollection that is also a cover.

Examples

(1) $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+2)$

has no finite subcover so

$\mathbb{R}$ is not compact.

(2) $[0,1]$ is <u>compact</u> (to be proven)

(3) $(0,1)$ is not compact

$(0,1) = \bigcup_{n \in \mathbb{Z}_+} \left(\frac{1}{n+2}, \frac{1}{n}\right)$

has no finite subcover.

$\boxed{\text{Theorems on compactness}}$

Th'm: X is compact, $f: X \to Y$ surjective and continuous $\Rightarrow$ Y is compact.

Proof Assume $Y = \bigcup U_\gamma$ open cover

$\Rightarrow X = \bigcup f^{-1}(U_\gamma)$ open cover since f is continuous.

$\Rightarrow \exists$ finite subcover by cptness $X = \bigcup_{i=1}^{n} f^{-1}(U_{\gamma_i})$

$\Rightarrow Y = \bigcup_{i=1}^{n} U_{\gamma_i}$ finite subcover. $\Rightarrow Y$ is cpt.