DEF: X is compact if every open cover has a finite subcover.

Example $X = [0,1]$ $\quad U_x = [0,1] \cap B_\varepsilon(x)$

open in X, $\quad X = \bigcup U_x$

$\Rightarrow \{ U_{n\frac{\varepsilon}{2}} : n = 0, \dots, \lceil \frac{2}{\varepsilon} \rceil \}$

ceiling

Using Ambient not relative open sets.

<u>Lemma</u>: $Y \subseteq \overline{X}$

Y is cpt $\Leftrightarrow$ every open cover of Y by sets open in $\overline{X}$ has a finite subcollection covering Y.

<u>Idea</u> of Proof: sets opening are $U \cap Y$ for U open in $\overline{X}$.

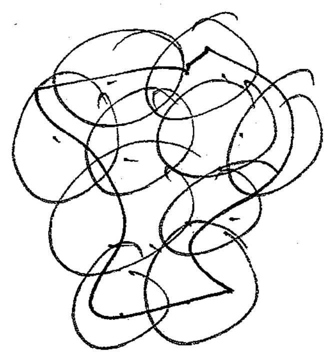
e.g. 1) $Y = [0,1] \subseteq \mathbb{R}$

$V_x = B_\varepsilon(x)$ open in $\underline{\underline{\mathbb{R}}}$ ~~$(x-\varepsilon, x+\varepsilon)$~~

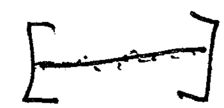
$\bigcup V_x \supseteq [0,1]$

$\Rightarrow \{ U_{n\frac{\varepsilon}{2}} : n = 0, \dots, \lceil \frac{2}{\varepsilon} \rceil \}$

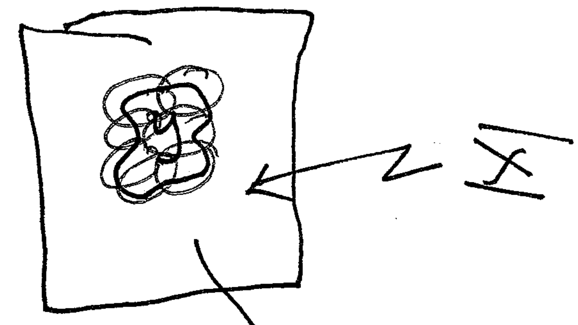
covers.



Thm Closed subspaces of cpt spaces are compact.



Pf: $Y \subseteq X$



$\mathcal{A}$ is a cover of Y by sets open in X.

Let $\mathcal{B} = \mathcal{A} \cup \{X - Y\}$ is an open cover of X since Y is closed.

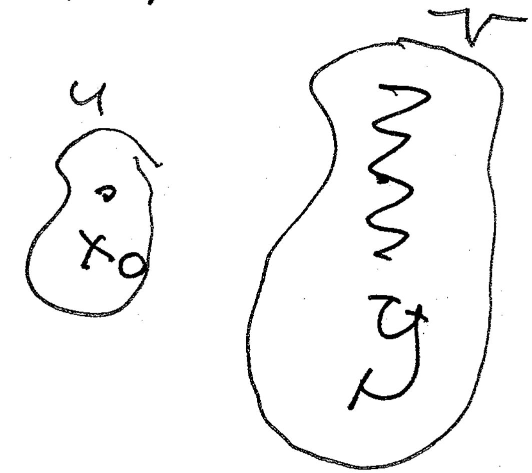
X is compact $\Rightarrow \mathcal{B}$ has a finite subcover.

~~Cre~~ Create a finite subcover of Y by throwing out $X - Y$ if it is in the finite subcover of $\mathcal{B}$. $\boxtimes$

Thm In HD spaces cpt subsets are closed.

Proof uses a Prelim Lemma.

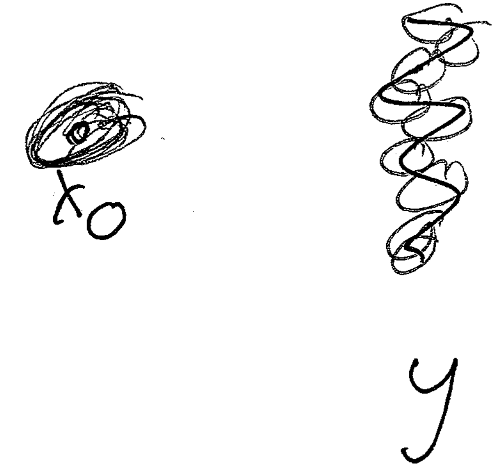
Prelim Lemma $\mathcal{Y} \subseteq X$, $\mathcal{Y}$ cpt, X HD



$x_0 \notin \mathcal{Y} \Rightarrow \exists$ open U, V
with $x_0 \in U$, $\mathcal{Y} \subseteq V$
and $U \cap V = \emptyset$.

Proof: $\forall y \in \mathcal{Y}$, $\exists$ open
U_y, V_y $\quad y \in V_y \quad x_0 \in U_y$
and $U_y \cap V_y = \emptyset$ by HD.

$\Rightarrow \bigcup V_y$ covers Y



$\Rightarrow$ cptness $\exists$ finite

subcover $V_{y_1} \cup \ldots \cup V_{y_n} \supseteq Y$

Let $V = V_{y_1} \cup \ldots \cup V_{y_n}$.

$U = U_{y_1} \cap \ldots \cap U_{y_n}$. is open.

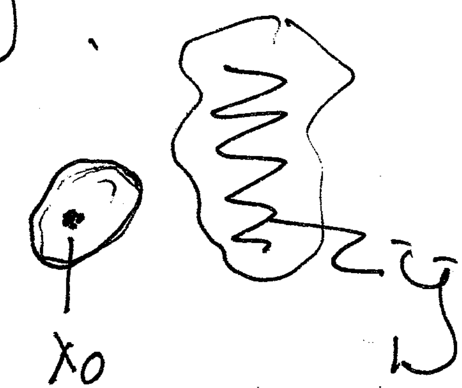
$x_0 \in U$, If $z \in V \Rightarrow z \in V_{y_i}$

$\Rightarrow z \notin U_{y_i} \Rightarrow z \notin U$. so

$U \cap V = \emptyset$. $\blacksquare$

Proof that cpt in HD $\Rightarrow$ closed.

For $x_0 \notin Y$ (Y cpt in HD X).



$\Rightarrow$ prelim lemma $\exists\, U$ with

$x_0 \in U$, $U \cap Y = \emptyset$

$\Rightarrow X - Y$ is open. So

Y is closed. $\blacksquare$

__Thm__: Cont image of compact space is compact.
(last time)

Th'm $f : X \to Y$ bijection, continuous

X cpt, Y HD $\Rightarrow$ f is ~~is~~ homeomorphism

or f^{-1} is cont

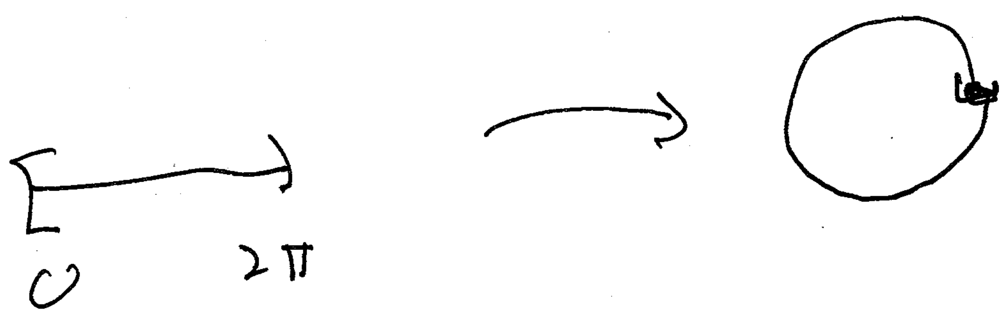
Proof f^{-1} is cont $\Leftrightarrow$ $(f^{-1})^{-1}$ closed sets are closed $\Leftrightarrow$ f(closed) = closed.

C closed in X $\Rightarrow$ C cpt $\Rightarrow$ $f(C)$ is cpt. $\Rightarrow$ Y HD, $f(C)$ is closed.

Example

$$f : [0, 2\pi) \longrightarrow S^1$$

$$f(t) = (\cos t, \sin t)$$



is continuous, bijection,

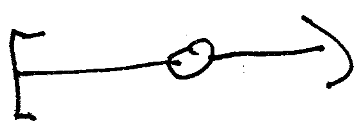
is it a homeomorphism?

<u>**No**</u> $[0, 2\pi)$ is not compact

Can $[0, 2\pi)$ be homeomorphic to S^1.

$\overset{\shortparallel}{I}$

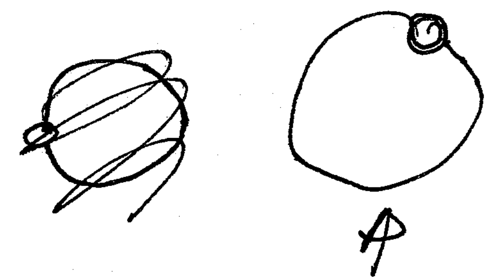
$I - \{\pi\}. \longrightarrow S^1 - pt.$



$\longrightarrow$

not conn

also homeomorphism



connected.

$$\boxed{\begin{array}{l} A \text{ homeo } B \\ A \text{ conn} \Leftrightarrow B \text{ conn} \end{array}}$$

Is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$

no

$\mathbb{R} - \{0\}$ is disconn.

$\mathbb{R}^2 - \{pt\}$ is path conn $\Rightarrow$ conn

$\boxed{\text{Products of cpt spaces are compact.}}$

in the product topology.

Tychonoff Thm