DEF: $X$ is compact if every open cover has a finite subcover.

Example: $X = \{0, 1\}$ \( \cup_{x \in \mathbb{R}} B_{\epsilon}(x) \) open

\( \cap_{x \in \mathbb{R}} X = U \cap X \)

\( \Rightarrow \exists \bigcup_{n=0}^{\infty} : n = 0, 1, 2, \ldots \) ceiling
Using Ambient not Relative open sets.

Lemma: \( y \subseteq \overline{X} \)

\( y \) is compact \( \iff \) every open cover of \( y \) by sets open in \( X \) has a finite subcollection covering \( y \).

Proof: Sets opening any \( U \cap y \) for \( U \) open in \( X \).
\[ y = \sum_{0,1} \in \mathbb{R} \]

\[ V_x = B_{\varepsilon}(x) \text{ open in } \mathbb{R} \]

\[ V V_x \supseteq \sum_{0,1} \]

\[ \Rightarrow \exists U_n \subseteq : n = 0, 1, \ldots \text{ covers.} \]
Theorem: Closed subspaces of cpt spaces are compact.

Proof: $y \subset \bar{x}$

A is a cover of $y$ by sets open in $X$.

Let $B = A \cup \exists x-y \bar{y}$ is an open cover of $X$ since $Y$ is closed.

$X$ is compact $\Rightarrow B$ has a finite subcover.
Create a finite subcover of \( y \) by throwing out \( X - y \) if it is in the finite subcover of \( B \). 

In HD spaces, open subsets are closed.

Proof uses a Prelim Lemma.
\[ \text{Proof: } A \neq \emptyset, E \text{ open} \]
\[ \emptyset \neq \bigcap_{y \in E} \bigcup_{x \in A} [x, y] \]
\[ x_0 \neq y \Rightarrow \exists \text{ open } U \]
\[ x_0 \neq y \Rightarrow U \nsubseteq E \cap I \]
 Suppose $V_y$ covers $y$.

 Let $V = V_{y_1} \cap V_{y_2} \cap \ldots \cap V_{y_n}$.

 Then $V \in \mathcal{N}_y$. Is open.

 So $y \in \mathcal{N}_y$, i.e., $y \notin U$. So $x_0 \in U$, i.e., $x_0 \in U \cup y$.
Proof that $\text{cpt in } H \implies \text{closed.}$

For $x_0 \not\in Y \ (Y \text{ cpt in } H \TEX X),$

$\implies \text{prelim lemma } E \cup x_0$

$x_0 \in U \cup \forall Y = \emptyset$

$\implies X - Y \text{ is open. }$ so

$Y \text{ is, closed. }$
Thin: Cost image is compact space is compact.
(last line)
If \( f: X \to Y \) is a bijection, continuous \( f \) is a homeomorphism:

\[ X \text{ is open} \iff Y \text{ is open} \iff f \text{ is a homeomorphism} \]

or \( f^{-1} \) is continuous.

**Proof:** \( f^{-1} \) is continuous \( \iff (f^{-1})^{-1} \) is continuous.

Closed sets are closed \( \iff f(C) = \text{closed} \).

Thus \( C \subseteq X \Rightarrow C \text{ is closed} \Rightarrow f(C) \text{ is closed.} \)
Example

\[ f: \Sigma_0, 2\pi \rightarrow S^1 \]

\[ f(\pm) = (\cos t, \sin t) \]

Is continuous, bijective?

Is it a homeomorphism?

No, \( \Sigma_0, 2\pi \) is not compact.
Can $S^0$, $2\pi i$ be homeomorphic to $S^1$?

$I \to I$

$I - 3 \pi i \to S^1 - \text{pt.}$

not conn  \quad \text{not homeomorphic}

\[
\begin{align*}
A \to B \quad \text{conn} \\
A \text{ conn } \iff B \text{ conn}
\end{align*}
\]

A connected.
Is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$?

No

$\mathbb{R} - 3$ is disconnect.

$\mathbb{R}^2$ is path connected.

$\Rightarrow$ connected.

Products of compact spaces are compact.

In the product topology, Tychonoff theorem.