DEF: \( C = \{ C_x : x \in \mathbb{N} \} \) has the finite intersection property if

\[ \bigcap_{l=1}^{n} C_L \neq \emptyset \]

\[ \forall n \in \mathbb{N} \]

all nonempty has f.i.p.

(1) \( \bigcap_{L=0}^{\infty} C_L \neq \emptyset \)

(2) \( \bigcap_{L=1}^{\infty} C_L = \emptyset \)

\( \left[ n^{-1}, n \right] = C_n \)

\( [0, 1/n] = C_n \).
Thm: \( X \) is c.p. \( \Rightarrow \) every collection \( \mathcal{C} \times \mathcal{C} \) of closed sets with the f.p. has \( \bigcap_{x \in X} C_x \neq \emptyset \)

Proof: Rewrite compactness: \( X \) is c.p.

Given a collection of open sets \( \mathcal{A} \), if \( \mathcal{A} \) covers \( X \) \( \Rightarrow \) \( \exists \) finite subcover.

Equivariant: Given a collection of open sets if no finite subcollection covers \( X \) \( \Rightarrow \) \( A \) doesn't cover.
We convert this into a statement about sets. Using these facts:

Given $A$, define $C_x = X - A_x$ and $C = \bigcap C_x \supseteq A_0$

(1) $A$ are all open $\iff$ $C$ are all closed.

(2) $A$ covers $X \iff \bigcap C_x = \emptyset$

(3) $A_1, \ldots, A_n$ covers $X \iff \bigcap C_x = \emptyset$.

De Morgan's Law
But these translate the alternating description of compactness to a statement about closed sets which is the theorem statement.

Example

\[ \overline{X} = (0, 1) \]

\[ C_n = (0, \frac{1}{n}] \text{ is closed in } \overline{X} \]

\[ C_n = [-1, \frac{1}{n}] \cap \overline{X} \]

\[ \bigcap C_n = \emptyset \implies \overline{X} \text{ is not compact.} \]
Theorem: $[a, b] \subseteq \mathbb{R}$ is compact

$-\infty < a < b < \infty$ (There is a more general theorem for strict linear orders with l.u.b. property)

Proof: Let $A$ is a covering of $[a, b]$

but sets open in the subspace topology.

We want a finite subcover.
Step 1: Claim! \( x \in [a, b] \), \( x \neq b \)

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Step 2: \( C = \{ y \in \mathbb{Q} : y > a \} \) and \([a, y]\) can be covered by finitely many elements of \( A \). By Step 1, \( C \neq \emptyset \). Let \( c = \text{Sup}(C) \) so we know \( a < c \leq b \). Goal now is to show \( c = b \).

Step 3: Claim: \( c \in C \). or that \([a, c]\) is covered by finitely many elements of \( A \).
Proof: claim!

$c \in \left[ q, b \right]$ \implies A', \forall A, A' with $c \in A'$. So, \exists d < c with \( (d, c] \subseteq A' \). Since $c = \sup C \implies \exists z \in (d, c] \cup C$ since $z \in C$, then $\left[ q, z \right]$ is covered by $s \equiv y$ $n$ elements of $A$. But $\Sigma z, c] \subseteq A' \implies$

$\Sigma q, c] = \Sigma q, z] \cup \Sigma z, c] \Rightarrow \text{ covered by finitely many }$

covered by $n$ covered by $A'$

c $\in C$. 

Step 4: \( C = b \). Assume to be contrary that \( C < b \).

By step 1, \( \exists y > c \) with

\[ \Sigma_c, y \] covered by \( \exists \) elemt of \( A \)

\[ \Sigma_0, y ] = \Sigma_0, c ] U [ c, y ] \]

\[ \text{covered by } \]

\[ \text{covered by } 1 \]

finite sup elemt

\( C \in \mathbb{C} \)

\[ \Sigma_0, y ] \text{ is covered by finitely many } \]

\[ \Rightarrow y \in C \]

\[ \text{Since } c = \text{ sup}(C) \]
\[ f \circ \phi \subseteq \phi \circ f \quad \text{with} \quad x \in \mathbb{R} \]

Every \( f : X \to \mathbb{R} \) is continuous \( X \) is compact.

\[ f \text{ is a continuous function} \]

Extreme Value Theorem.

If it is closed and bounded, \( f \) is compact.

Heine-Borel: \( C \subseteq \mathbb{R} \) is compact.
Lemma (Prelim).

If \( C \subseteq \mathbb{R} \) is compact

\[
\Rightarrow \sup(c) \in C \\
\inf(c) \in C
\]

Proof: We know \( C \) is bounded

by \( H-B \), so \( \sup(c) < \infty \), \( \inf(c) > -\infty \).

But \( C \) is closed, so \( C = \mathbb{R} - C \) is open.

So \( \sup(c) \in C \Rightarrow \exists \)

\[
(c - \varepsilon, c + \varepsilon) \subseteq \mathbb{R} - C
\]

\[
\Rightarrow c - \varepsilon \text{ is is as smaller upper bound}
\]
Proof of EVT:

$f(X)$ is cpt so by HB close and bounded and by Lemma

\[ \sup_{x \in X} f(x) \leq f(X) \]
\[ \inf_{x \in X} f(x) \leq f(X) \]

\[ \implies \exists c \text{ with } f(c) = \sup_{x \in X} f(x) \]
\[ f(c) = \inf_{x \in X} f(x) \]

\[ f(c) \leq f(x) \leq f(c) \]
$f : S^2 \to \mathbb{R}$

$S^2 = \{ x \in \mathbb{R}^3 : ||x|| = 1 \}$

Image - connected and closed and bounded

Assuming Temp cont.