DEF: $\mathcal{C} = \{C_\lambda : \lambda \in \Lambda\}$ has the finite intersection property if $\forall\, C_1, \dots, C_n \in \mathcal{C}$

$$\bigcap_{i=1}^{n} C_i \neq \emptyset$$

eg// $C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$ all nonempty

has f.i.p.

(1) $\bigcap_{i=\emptyset}^{\infty} C_i \neq \emptyset$ $\quad [-\frac{1}{n}, \frac{1}{n}] = C_n$

(2) $\bigcap_{i=1}^{\infty} C_i = \emptyset$ $\quad (0, 1/n] = C_n$.

Th'm: X is cpt $\iff$ every collection $\{C_\lambda\}$ of closed sub sets with the f.i.p. has $\bigcap_{\lambda \in \Lambda} C_\lambda \neq \emptyset$

Proof: Rewrite compactness: ~~X~~ X is cpt

Given $\mathcal{A}$ collection of open sets if $\mathcal{A}$ covers $X \Rightarrow \exists$ finite subcover.

equivalent: [Given a collection of open sets if no finite sub collection covers then $\mathcal{A}$ doesn't cover.]

We convert this into a statement about sets. Using these facts

Given $\mathcal{A}$, define $C_\lambda = X - A_\lambda$ and $\mathcal{C} = \{C_\lambda\}$

(1) $\mathcal{A}$ ~~are~~ are all ope $\iff$ $\mathcal{C}$ are all closed.

(2) $\mathcal{A}$ covers X $\iff$ $\bigcap C_\lambda = \emptyset$

(3) $A_1, \ldots, A_n$ covers X $\iff$ $\bigcap_{i=1}^{n} C_i = \emptyset$.

$\boxed{\text{De Morgan's Law}}$

But these translate the alternative description of compactness to a statement about closed sets which is the theorem statement. ∎

<u>example</u> $\boxed{X = (0,1)}$

$C_n = (0, 1/n]$ is closed in $\underline{\underline{X}}$

$C_n = [-1, 1/n] \cap X$.

$\bigcap C_n = \emptyset \Rightarrow X$ is not cpt.

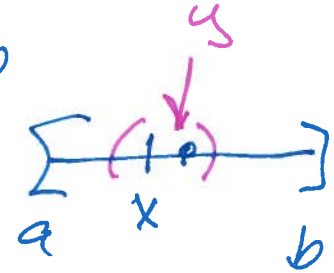
order topology = usual topology

Th'm: $[a,b] \subseteq \mathbb{R}$ is compact

$-\infty < a < b < \infty$ (There is a more general theorem for strict linear orders with l.u.b. property)

Proof Let $\mathcal{A}$ is a cover of $[a,b]$ by sets open in the subspace topology. We want a finite subcover.

Step 1: **Claim:** $x \in [a,b]$, $x \neq b$



$\Rightarrow \exists\, y > x$ such that

$[x,y]$ can be covered by one A_i

Pf. $\exists\, A_{i_0} \in \mathcal{A}$ with $x \in A_{i_0}$. Since

A_c open in order topo $\Rightarrow \exists\, c \neq x$

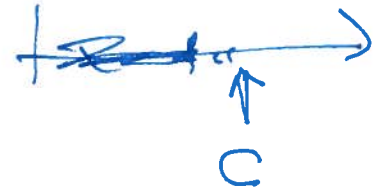
with $[x,c) \subseteq A_{i_0}$. Pick $y \in [x,c)$

Then $[x,y] \subseteq A_{i_0}$.

Step 2: $G = \{ y \in [a,b] : y > a$ and $[a,y]$ can be covered by finitely many elements of $A \}$. By step 1, $G \neq \emptyset$. Let $c = \sup(G)$ so we know $a < c \leq b$. Goal now is to show $c = b$.

Step 3 Claim: $c \in G$. or that $[a,c]$ is covered by finitely many elements of A.

Proof of claim:

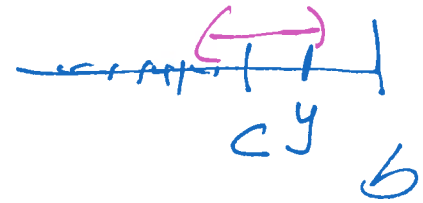


$c \in [a,b] \Rightarrow A' \in A$.

with $c \in A'$. So $\exists\, d < c$ with $(d, c] \subseteq A'$. Since $c = \sup C \Rightarrow \exists\ z \in (d,c] \cap C$. Since $z \in C$, then $[a,z]$ is covered by some n elements of A. But $[z,c] \subseteq A' \Rightarrow$

$$[a,c] = \underbrace{[a,z]}_{\text{covered by } n} \cup \underbrace{[z,c]}_{\text{covered by } A'} \Rightarrow \text{covered by finitely many } c \in C.$$

Step 4: $c = b$. Assume to be contrary that $c < b$.



By step 1, $\exists\ \underline{y > c}$ with

$[c, y]$ covered by 1-element of A

$$[a, y] = \underbrace{[a, c]}_{\substack{\text{covered by} \\ \text{finite sing} \\ c \in G}} \cup \underbrace{[c, y]}_{\substack{\text{covered} \\ \text{by 1} \\ \text{element}}}$$

$[a, y]$ is covered by finitely many $\Rightarrow y \in G$ $\times$

$\boxed{y > c \quad \text{since } c = \mathrm{Sup}(G)}$

Heine-Borel: $C \subseteq \mathbb{R}^n$ is compact

$\Longleftrightarrow$ it is closed and bounded.

Extreme Value Theorem.

$f: X \to \mathbb{R}$ continuous X is compact

$\Rightarrow \exists\, c, d \in X$ with.

$$f(c) \leq f(x) \leq f(d)$$

$$\forall x \in X.$$

Lemma (Prelim).

If $C \subseteq \mathbb{R}$ is compact

$\Rightarrow$ $\sup(C) \in C$

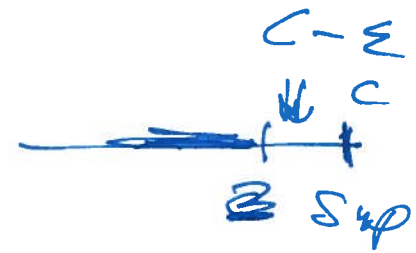
$\inf(C) \in C$

<u>Proof</u>: We know C is bounded by H-B, so $\sup(C) < \infty$, $\inf > -\infty$.

But C is closed, so $\mathbb{R} - C$ is open

so $\sup(C) \notin C$ $\Rightarrow$ $\exists$

$(c-\varepsilon, c+\varepsilon) \subseteq \mathbb{R} - C$



$\Rightarrow$ $c-\varepsilon$ is is a smaller upper bound $\times$

Proof of IVT:

$f(x)$ is continuous on $[a,b]$

closed and bounded

Lemma 1

$|f(x)| \le M$

$|f(x)| \le M$

$|f(x)| \le M$

$|f(x)| \le M$

$\Downarrow$

$\Downarrow$

$f(c) = \sup f(x)$

$f(c) = \inf f(x)$

$f(c) = \sup f(x)$

$f(c) = \inf f(x)$

So by HB

$f(c) = 0$

$\blacksquare$

$f: S^2 \to \mathbb{R}$

$S^2 = \{ \vec{x} \in \mathbb{R}^3 : \|\vec{x}\| = 1 \}$

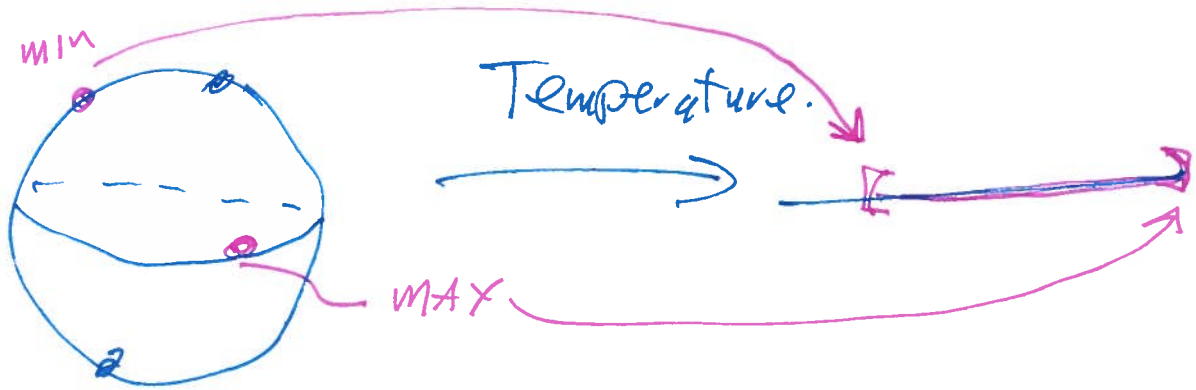


Image — connected and closed and bounded

Assuming Temp cont.