CORR OF FIP property of compact sets

COR: X cpt $C_1 \supset C_2 \supset C_3 \supset \dots$
nested family of $\boxed{\text{closed}}$ nonempty

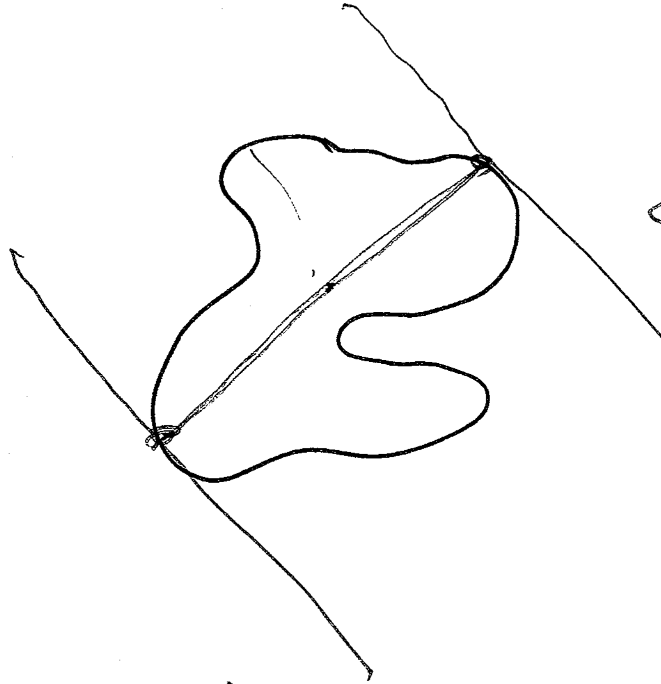
$$\Rightarrow \bigcap_{i=1}^{\infty} C_i \neq \emptyset$$

Applications of compactness in metric spaces.

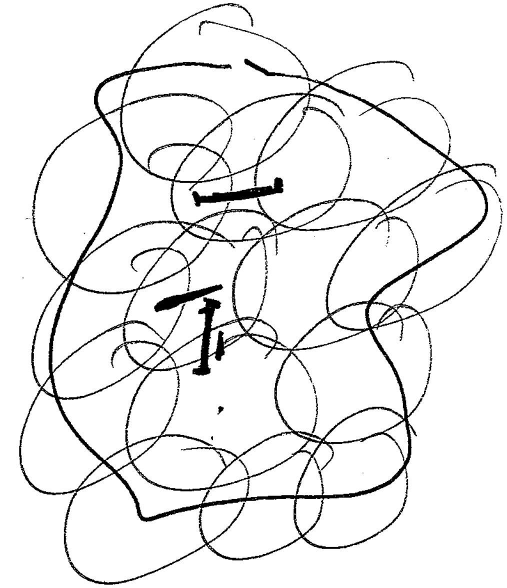
DEF: (1) $A \subseteq (X,d)$, $d(x,A) := \inf\{d(x,a) : a \in A\}$
(we showed this continuous)

(2) $A \subseteq (X,d)$, $\operatorname{diam}(A) := \sup\{d(a_1,a_2) : a_1,a_2 \in A\}$

In $\mathbb{R}^2$



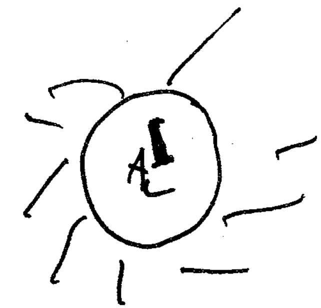
diam



$\operatorname{diam}([0,\infty)) = \infty.$

<u>Lebesgue number lemma</u>: $\mathcal{A}$ is an open cover of a compact, metric space X $\Rightarrow$ $\exists\, \delta > 0$ (the Leb. #) so that $Y \subseteq X$, $\operatorname{diam}(Y) < \delta \Rightarrow \exists\, A \in \mathcal{A}$ with $Y \subseteq A$.

Proof: If $X \in A$ done, so assume $X \notin A$. Pass to finite subcover $A_1, \ldots, A_n$. Let $C_i = X - A_i$



Define $f : X \to \mathbb{R}$

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

This is continuous. Claim: $f(x) > 0 \ \forall x$.

Proof of claim: $x \in A_i$ for some $A_i \Rightarrow$

$\exists \varepsilon > 0$ with $B_\varepsilon(x) \subseteq A_i$

$\Rightarrow d(x, C_i) \geq \varepsilon$

$\Rightarrow f(x) > \varepsilon / n$.

Since $\mathbb{X}$ is cpt, extreme value th'm says $\exists\ \delta = \min\{f(x) ; x \in \mathbb{X}\} > 0$

Since $f(x) > 0$

We now show that δ is the Leb #.

Proof of this: Assume $\operatorname{diam}(Y) < \delta$

$x_0 \in Y \Rightarrow Y \subseteq B_\delta(x_0)$.

x_0.

$\delta < f(x_0) \leq \max d(x_0, C_i)$

$= d(x_0, C_m)$ some m.

Thus $Y \subseteq B_\delta(x_0) \subseteq \mathbb{X} - C_m = A_m$. ∎

Uniform Continuity

continuity ~~at x~~

$\underline{\forall x \in X}, \forall \varepsilon > 0, \exists \delta > 0$ so that

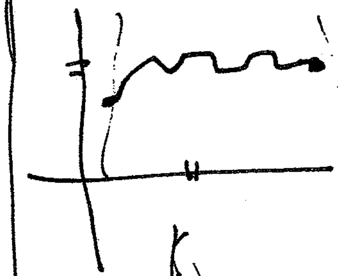
$d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$

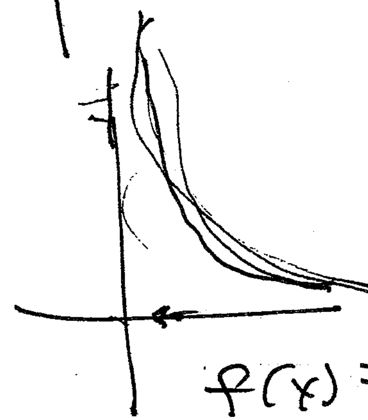
Uniform continuity.

$\forall \varepsilon > 0, \exists \delta > 0, \underline{\forall x \in X}$

$d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$

one δ works for all $\underline{x}$.





$f(x) = 1/x$

$\boxed{\text{given } x, \varepsilon \Rightarrow \exists \delta(x)}$

$x \to 0, \delta(x) \to 0$

non uniform continuity.

Theorem: $f : (X, d_X) \to (Y, d_Y)$

is continuous and X is compact

$\Rightarrow f$ is uniformly continuous.

Proof: Given $\varepsilon > 0$, $\{ B_{\varepsilon/2}(y) : y \in Y \}$

is open cover of Y. Let

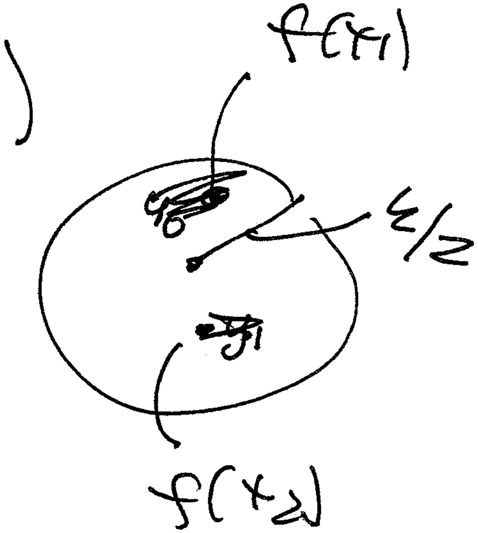
$$\mathcal{A} = \{ f^{-1}(B_{\varepsilon/2}(y)) : y \in Y \}$$

is an open cover of X by continuity.

Let δ be its Lebesgue number.

Pick $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$ then $\{x_1, x_2\}$ has $\operatorname{diam}(\{x_1, x_2\}) < \delta$. $\Rightarrow$ by the Leb # Lemma, $\exists\, y_0$ with

$$\{x_1, x_2\} \subseteq f^{-1}(B_{\varepsilon/2}(y_0))$$



$$f(x_1) \in B_{\varepsilon/2}(y_0)$$
$$f(x_2) \in B_{\varepsilon/2}(y_0)$$

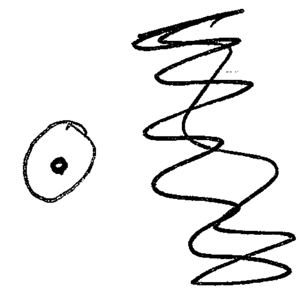
$$\Rightarrow d(f(x_1), f(x_2)) < \varepsilon. \quad \blacksquare$$

Back to general HD spaces.

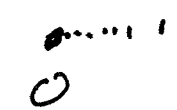
DEF: (1) $x \in X$ is called <u>isolated</u> if $\{x\}$ is an open set.

(2) If $Y \subseteq X$, $y_0 \in Y$ is <u>isolated</u>

$\Leftrightarrow$ $\exists$ open U with $y_0 \in U$ and $U \cap Y = \{y_0\}$



Example $\{1/n : n \in \mathbb{Z}_+\} \cup \{0\}$.



$\to$ cpt, $\{0\}$ is not isolated and each $\{1/n\}$ is isolated. countable.

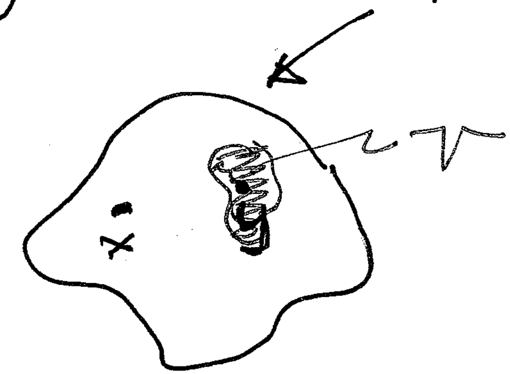
Th'm X is a nonempty compact HD space with no isolated pts $\Longrightarrow$ X is uncountable

Example $\mathbb{Q}$ countable, HD, no isolated point, not compact.

COR: $[a,b]$ $a \neq b$ is uncountable and any $Y \subseteq \mathbb{R}$ which contains a nontrivial closed interval is uncountable.

e.g. $(0,1) \supseteq [.1,.9]$

Step 1: claim: Given open $U \neq \emptyset$, $x \in X \Rightarrow \exists$ open, non-empty $V \subseteq U$ with $x \notin \overline{V}$.



Proof of claim: Pick $y \in U$ with $y \neq x$, possible since x is not isolated and $U \neq \emptyset$.

Use HD. Find nbhds W_1 of x and W_2 of y with $W_1 \cap W_2 = \emptyset$. Let

$V = U \cap W_2$ open, $y \in V$ nonempty.

$V \subseteq U$ and $x \notin \overline{V}$, because $x \in \overline{V}$ ~~since $x \notin \overline{V}$~~ would mean every open set that contains x would contain a point of V but W_1 doesn't. ∎