$\boxed{\text{complete metric spaces}}$ Top 29

(X, d) is metric

DEF: $\{x_n\} \subseteq X$ is called a Cauchy sequence if $\forall \varepsilon > 0 \; \exists N$ so that $n, m > N \Longrightarrow$

$$d(x_n, x_m) < \varepsilon$$

Tells how close elements of the sequence are to each other not how close to limit.

__DEF:__ (X, d) is __complete__ if every Cauchy sequence converges.

__Remarks__

(1) $\{x_n\}$ conv $\Rightarrow$ Cauchy

(2) Closed subsets of complete spaces are complete

(3) $\boxed{\mathbb{Q} \cap [0,1]}$ not complete. Cauchy seq has converge to irrationals

(4) $\bar{d}$ is the bounded metric derived from d. $(X, \bar{d})$ is complete $\Leftrightarrow$ (X, d) is complete.

Lemma (X, d) is complete $\Leftrightarrow$ every Cauchy seq. has a convergent subsequence.

Proof ($\Rightarrow$) TRIVIAL. ($\Leftarrow$) Says is that if a Cauchy seq has a conv subseq, it converges.

Assume $\{x_n\}$ is Cauchy with $\{x_{n_i}\}$ convergent subsequence with $x_{n_i} \to x$. Given $\varepsilon > 0$

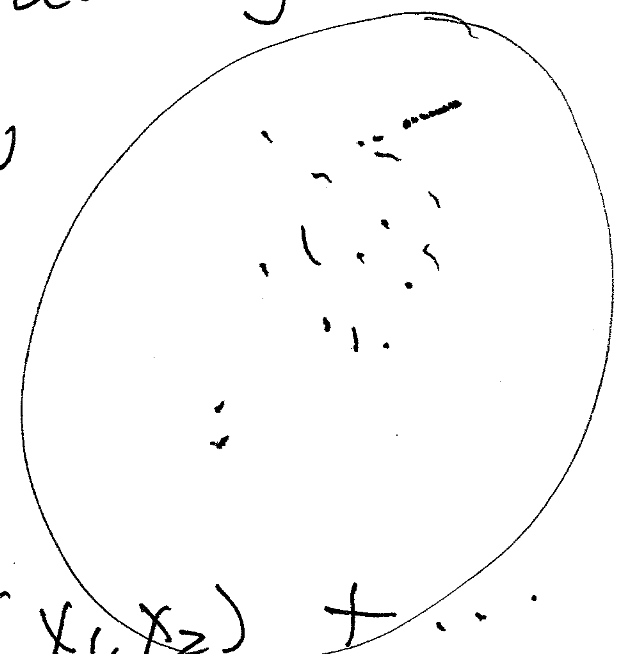
1 Since $\{x_n\}$ is Cauchy $\exists$ N, $n,m > N \Rightarrow d(x_n, x_m) < \varepsilon/2$.

2 Pick i large enough so that $n_i > N$ and $d(x_{n_i}, x) < \varepsilon/2$. Then $\boxed{n \geq N}$

$$\boxed{d(x_n, x) \leq d(x_n, x_{n_i}) + d(x_{n_i}, x) \leq \underset{1}{\varepsilon/2} + \underset{2}{\frac{\varepsilon}{2}} < \varepsilon.} \blacksquare$$

Thm $(\mathbb{R}^k, d)$ is complete.

Proof: $\{x_n\}$ is Cauchy



Pick N so that $\forall n, m > N$

$\Rightarrow d(\cancel{x_n}, x_m) < 1$

When $k > N+1$

$$d(x_k, 0) \leq d(0, x_1) + d(x_1, x_2) + \cdots + d(x_{N-1}, x_N) + d(x_{N+1}, x_k)$$

$$< d(0, x_1) + d(x_1, x_2) + \cdots + d(x_{N-1}, x_N) + 1$$

$$= M.$$

So $d(x_i, 0) < M \quad \forall i$

So $\{x_n\} \subseteq \overline{B_M(0)}$

which is compact and

$\{x_n\}$ is infinite (if finite is trivial)

so has a limit point y so $\exists$

a sequence $\{x_{n_i}\} \subseteq \{x_n\}$ with

with $x_{n_i} \to y$. By the previous

$x_n \to y$. $\blacksquare$

$\boxed{\text{RECALL}}$ $(\mathbb{R}^{\mathbb{Z}^+}, D)$

$$D(\underline{x}, \underline{y}) = \sup_{i \in \mathbb{Z}^+} \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\}$$

..yields product topology.

Theorem $(\mathbb{R}^{\mathbb{Z}^+}, D)$ is complete.

Proof: $\underline{X}^{(n)}$ is Cauchy

Since $\bar{d}(\pi_i(\underline{x}), \pi_i(\underline{y})) \leq i\, D(\underline{x}, \underline{y})$

$\Rightarrow$ $\pi_i(\underline{X}^{(n)})$ is Cauchy for each i

Thus for each i, $\pi_i(x^{(n)}) \to y_i$

Since $\mathbb{R}$ is ~~Cauchy~~ complete $\Rightarrow$

$\boxed{\text{HW}}$ $x^{(n)} \to \underline{y}$. $\blacksquare$

What about $\mathbb{R}^J$ larger J?

Problem with product topology.

But uniform metric is complete.

$\underline{x}, \underline{y} \in \mathcal{Y}^{\Lambda}$ are Λ-tuples.

so $\underline{x} = (x_\lambda)_{\lambda \in \Lambda}$

OR $\underline{x}$ is a function

$f : \Lambda \to \mathcal{Y}$.

$$\mathbb{R}^{\mathbb{Z}_+} = \{ f : \mathbb{Z}_+ \to \mathbb{R} \}$$

Uniform or sup metric

$$\bar{\rho}(\underline{x}, \underline{y}) = \sup\{\bar{d}(x_\lambda, y_\lambda) : \lambda \in \Lambda\}$$

where $\bar{d}$ is bounded metric coming from

(Y, d)

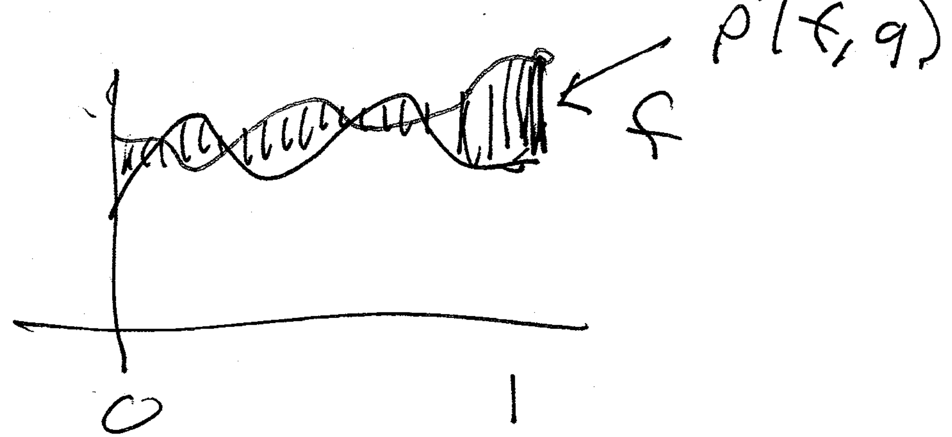
<u>OR</u>

$$\bar{\rho}(f,g) = \sup\{\bar{d}(f(\lambda), g(\lambda)) : \lambda \in \Lambda\}$$

<u>Example</u>: $\mathcal{F} = \{f : [0,1] \to \mathbb{R}\} = \mathbb{R}^{[0,1]}$

$$\bar{\rho}(f,g) = \sup\{\sigma|f(x) - g(x)| : x \in [0,1]\}$$

$\sigma(\omega) = \min(\omega, 1)$



<u>Th'm</u>: If (Y, d) is complete $\Rightarrow Y^{\underline{\underline{I}}}$ is complete in the uniform metric.

<u>Proof</u>: ~~P~~ Show components are Cauchy etc. (See book)

Now $\Lambda = X$ be a topological space. $\boxed{Y = (Y, d) \text{ metric}}$

$Y^X = \{ f : X \to Y \}$

$\mathcal{C}(X, Y) \subseteq Y^X$

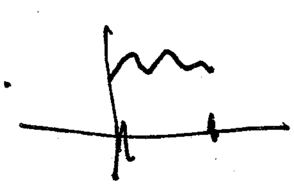
$\mathcal{C}(X, Y) = \{ f : X \to Y : f \text{ is continuous} \}$

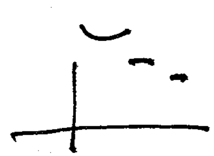
$\mathcal{B}(X, Y) = \{ f : X \to Y : f \text{ is bounded } \operatorname{diam} f(X) < \infty \}$

$\mathcal{B}(X, Y) \subseteq Y^X.$

$X = [0,1]$ $Y = \mathbb{R}$

X cpt $\mathcal{C}(X, Y) \subseteq \mathcal{B}(X, Y)$





Theorem $(X, \mathcal{T})$, (Y, d) complete metric

$\Rightarrow \mathcal{C}(X, Y)$ and $\mathcal{B}(X, Y)$ are closed in Y^X with the uniform metric and so are complete.

Important idea in proof

$f_n \in \mathcal{C}(X, Y)$

$f_n \to f$ in Y^X unif metric

$\Rightarrow f$ is continuous.