Using last time.

\( C(X, (Y, d)) \) with \( X \) compact, \((Y, d)\) complete.

is complete with "Sup metric".

\[ d(f, g) = \sup \left\{ d(f(x), g(x)) : x \in X \right\} \]

Example: Space Filling Curve. \( \Gamma = \Sigma 0,1 \)

\( \alpha : I \rightarrow I^2 \)

Continuous and onto.

Not injective.
Idea: Define a sequence

\[ f_n \in C(I, I^2) \]

such that \( f_n \) converges in some manner to \( f \) from previous proofs.

Then \( f_n \) is Cauchy.

Thus \( f_n \to f \) since \( C(I, I^2) \) is complete and \( f \) is the space filling curve.
Special Features of Complete Metric Spaces.

Theorem \((\mathcal{X}, d)\) is compact \(\iff\) complete and totally bounded.

**DEF.** \((\mathcal{X}, d)\) is totally bounded if \(\forall \varepsilon > 0, \exists\) finite cover of \(\mathcal{X}\) by \(\varepsilon\)-balls.
(1) Totally Bounded \[ \Rightarrow \] Bounded.

Proof: If \( X \) is covered by \( 2 \) \(-\) balls then

\[ d_{1n}(X) < 2N \]

(2) Converse is false.

\[ d_{1n}(IR) = 1 \]

\[ d_{1n}(\mathbb{R}) = \infty \]

\[ d_{1n}(\mathbb{Q}) \]

\[ P \]

\[ P \]

\[ P \]

Is bounded but not totally bounded.
\[ \lim_{n \to \infty} c_p^t = \text{complete } \pm 1, b. \]

\[
(\Rightarrow) \quad \exists x_3 \in X \text{ is Cauchy (and infinite)} \Rightarrow \text{ by compactness (sequence)}
\]

\[ \exists x_{n3} \subseteq X \text{ with } \exists \]

\[ \exists x_{n3} \text{ convergent by Lemma } \exists x_{n3} \text{ conv. } \]

\[ \exists x_{n3} \text{ conv. then by Lemma } \exists x_{n3} \text{ conv. } \]

\[ \text{So } X \text{ is complete. } \]

Now for \( \pm 1, b \), \( \exists B_\varepsilon (x) \) is an open cover of \( X \Rightarrow \) finite subcover by compactness \( \Rightarrow \pm 1, b. \)
(\rightarrow) Assume \( X \) is complete, i.e., we show \( X \) is seq. compact.

Given \( \exists (x_n) \) (not a finite set)

\[ \exists \{x_{n_k}\} \text{ is a subsequence of } (x_n) \]

Claim: \( \exists (x_{n_k}) \) Cauchy and thus \( \exists x \in X \) Cauchy and thus \( \exists x_{n_\infty} \) convergent. \( \Rightarrow X \) is seq. compact.

\( x_{n_\infty} \)
Proof of claim

Cover $X$ by finitely many $\mathcal{B}$-balls. One of these, say $B_{\frac{1}{2}}(q_1)$ contains infinitely many points of $\mathcal{E}^n$. Let

$$J_1 = \{ n \mid x_n \in B_{\frac{1}{2}}(q_1) \}.$$

Now balls of radius $\frac{1}{2}$, $\mathcal{B}_{\frac{1}{4}}(q_2)$ contains $\infty$ many pts of $\mathcal{E}^n$. Let

$$J_2 = \{ n \mid x_n \in B_{\frac{1}{4}}(q_2) \}.$$

Balls of radius $\frac{1}{2^{k+1}}$, $\mathcal{B}_{\frac{1}{2^{k+1}}}(q_2)$ contains $\infty$ many $J_k$: $n \in J_k$ $x_n \in B_{\frac{1}{2^{k+1}}}(q_2)$. 

$$J_3 = \{ n \mid x_n \in B_{\frac{1}{2^{k+1}}}(q_2) \}.$$
Now \( J_1 \leq J_2 \leq J_3 \leq \cdots \).

Each is infinite.

Pick \( n_1 \in J_1 \), \( n_2 > n_1 \), \( n_2 \in J_2 \), \( n_k \in J_k \), \( n_{k+1} > n_k \).

Then \( \forall i, j \geq k \quad x_{n_i} x_{n_j} \in B_{\frac{1}{k}}(a_k) \)

\( \Rightarrow \quad d(x_{n_i}, x_{n_j}) < \frac{1}{k} \)

\( \Rightarrow \quad \exists K \geq \frac{1}{k} \) is Cauchy.
DEF: \( f^* : (X,d) \to (X,d) \) self map

\[ f^* (x,d) \]

is a contraction if \( 0 < k < 1 \)

So that \( d(f(x), f(y)) \leq kd(x,y) \)

Iteration of functions

\[ f^2(x) = f(f(x)) = f\circ f(x) \]

\( n \geq 1 \) iterates

\[ f^n(x) = f(f(\ldots f(x)\ldots)) = f_0 \circ f \circ \ldots \circ f(x) \]
Contraction Mapping Theorem

\[ f: (\mathbb{R}, d) \rightarrow (\mathbb{R}, d) \text{ is a contraction and } (\mathbb{R}, d) \text{ is complete } \Rightarrow \]

1. \( f \) has a fixed point \( f(x_0) = x_0 \).
2. \( x_0 \) is the unique fixed point.
3. \( \forall x \in \mathbb{R}, \ f^n(x) \rightarrow x_0 \).

Proof

Prelim Lemma.

\[ \forall x, \ d(f^n(x), f^{n+1}(x)) \leq k^n d(x, f(x)) \]
Proof of Prelm by Induction:

\[ n = 1: d(f(x), f^2(x)) \]
\[ = d(f(f(x)), f^3(x)) \]
\[ \leq k d(x, f(x)) \]

Assume true for \( n \).

\[ d(f^{n+1}(x), f^{n+2}(x)) \]
\[ = d(f(f^n(x)), f(f^{n+1}(x))) \]
\[ \leq k d(f^n(x), f^{n+1}(x)) \]
\[ \leq k^{n+1} d(x, f(x)) \text{ by inductive hyp.} \]