Contraction Mapping Theorem

$f : (X, d) \circlearrowleft$, contraction ($\exists\ 0 < \lambda < 1$ w/m $d(f(x), f(y)) < \lambda\, d(x,y)\ \forall x, y \in X$)

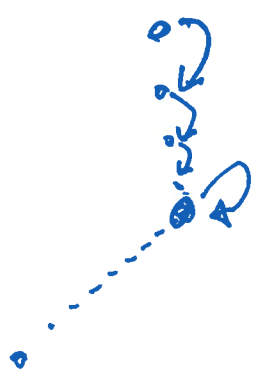
(X, d) complete $\Longrightarrow$

(1) f has a fixed point x_0, $f(x_0) = x_0$

(2) x_0 is a unique fixed theorem.

(3) $\forall x$, $f^n(x) \to x_0$ where

$f^n = f \circ \cdots \circ f$ (n-times)



global attractor

Prelim Lemma $\forall x \in X$

$$d(f^n(x), f^{n+1}(x)) \leq K^n d(x, f(x))$$

Claim $\{f^n(x)\}$ is Cauchy.

so $\sum_{m,n} \phi_m(x)\phi_n(x)$

$$= \sum_{n=0}^{\infty} \frac{1}{\lambda_n}\phi_n(x)\int \phi_n(x')\rho(x')\,dx'$$

$$= \int \rho(x')\sum_{n=0}^{\infty} \frac{1}{\lambda_n}\phi_n(x)\phi_n(x')$$

$$= \int \rho(x')\,G(x,x')$$

$$= \int \rho(x')\,\phi_1(x)\phi_1(x') + \int \rho(x')\,\phi_2(x)\phi_2(x') + \cdots + \int \rho(x')\,\phi_m(x)\phi_m(x') + \int \rho(x')\,\phi_{m+1}(x)\phi_{m+1}(x') + \cdots$$

problem

let $n < m$

Since (X, d) is complete, $\exists\, x_0$ with $f^n(x) \to x_0$ Since

f is <u>continuous</u>

$f^{n+1}(x) = f(f^n(x)) \to f(x_0)$

But $\{f^{n+1}(x)\} \subseteq \{f^n(x_0)\}$

seq conv.

so $f^{n+1}(x) \to x_0$

$\Rightarrow$ #1 $\boxed{x_0 = f(x_0)}$ by uniqueness of limits.

(2) Say $f(x_0) = x_0$ $\quad f(x_0') = \cancel{f(x_0)}\ x_0'$

?

So

$$d(x_0, x_0') = d(f(x_0), f(x_0'))$$
$$\leq K\, d(x_0, x_1)$$

contradiction to $K < 1$

(3) Since $\forall x$, $f^n(x)$ conv to a fixed point and there is just one of them, (3) follows. $\blacksquare$

<u>example</u> $f: \mathbb{R} \to \mathbb{R}$, continuously differentiable and $\exists\, k<1$ with

$$|f'(x)| \le k < 1 \quad \forall x \in \mathbb{R}$$

$\Longrightarrow$ unique fixed point.

<u>Proof</u> Mean Value Theorem. $\leftarrow$ ~~DV~~

$x \quad y$

$\forall\, x, y \quad x<y \quad \exists\, c \in (x,y)$

with $f'(c) = \dfrac{f(y)-f(x)}{y-x}$

$$|f(y)-f(x)| = |f'(c)|\,|y-x| \le k\,|y-x|$$

or $d(f(y), f(x)) \le k\, d(y,x)$.

example $f(x) = x+2$, on graph

$\boxed{f(f) = 1}$

$y = f(x) = x$ is the graph of f

$x = 0$ hits

Another important property of complete metric spaces

Baire Category Theorem

no relation to "Category Theory"
no relation to "Bears".
Bare

DEF

- $A \subseteq X$ is nowhere dense (closed) if $\boxed{\mathrm{Int}(A) = \emptyset}$.

- X is a Baire space if: when $\{A_n\}$ is a collection of nowhere dense closed sets $\Rightarrow \bigcup A_n$ has empty interior.

Example: In a Baire space (T_1) countable sets have empty interiors.

We will prove $\mathbb{R}^n$ is Baire.

So a countable set in $\mathbb{R}^n$ has empty interior.

Another formulation which equivalent

<u>Lemma</u>: X is Baire if whenever $\{U_n\}$ is a countable collection of open, dense sets $\Rightarrow$

$$\bigcap_{n \in \mathbb{Z}_+} U_n \text{ is dense.}$$

<u>example</u> $X = \mathbb{R}$ (a Baire space)

Let $\{r_1, \ldots\}$ be an enumeration of $\mathbb{Q}$. Let $U_n = \mathbb{R} - \{r_1, \ldots, r_n\}$.

It is open dence $\Rightarrow$

irrationals $= \bigcap_{n \in \mathbb{Z}_+} U_n$ is dense.

<u>Note:</u> The Theorem often uses the language of 1st category and 2nd category sets which we won't use.

Baire Category Theorem: If X is complete metric or compact Hausdorff $\Rightarrow X$ is Baire.

Proof needs a bit of machinery from later in the course.

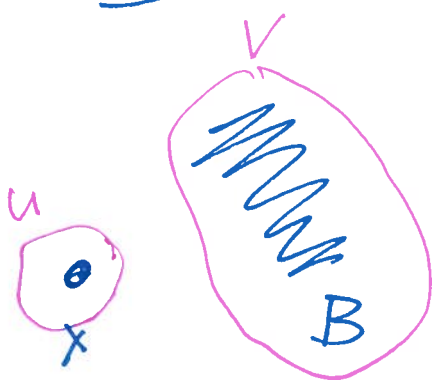
DEF: $(X, \mathcal{T})$ is regular if

(1) T_1 (points are closed)

(2) $\forall$ points $x \in X$ all closed sets B

$\exists\, U, V$ open $U \cap V = \emptyset$

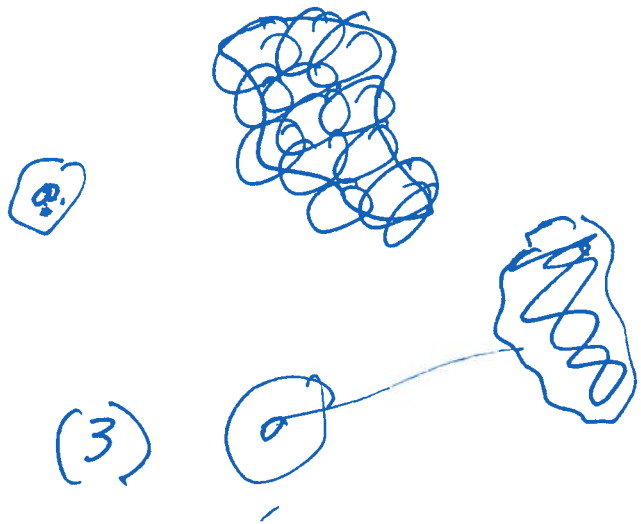
$x \in U \quad B \subseteq V$



examples

(1) cpt HD $\Rightarrow$ regular

(2) metric $\Rightarrow$ regular.



(3)

$\overline{\mathrm{Int}(A)} \neq A$

$d(x, B) > 0$

since $x \notin \bar{B} = B$.