Contraction Mapping Theorem

\( f : (X, d) \rightarrow (Y, d), \) contraction \( (\exists \alpha < 1 \text{ s.t. } \forall x, y \in X, d(f(x), f(y)) < \alpha d(x, y) ) \)

\( (X, d) \) complete \( \Rightarrow \)

1. \( f \) has a fixed point \( x_0, f(x_0) = x_0 \)
2. \( x_0 \) is a unique fixed point.
3. \( \forall x, f^n(x) \Rightarrow x_0 \) where

\[ f^n = f \circ \ldots \circ f \text{ (n-times)} \]

global attraction
Claim \( \exists f_n(x)^3 \) is Cauchy.
Let $m > n$

\[ d(f^n(x), f^m(x)) \leq d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), f^m(x)) \]

\[ + \cdots + d(f^{m-1}(x), f^m(x)) \]

\[ \leq k^n d(x, f(x)) + k^{n+1} d(x, f(x)) \]

\[ + \cdots + k^{m-1} d(x, f(x)) \]

\[ \leq (x, f(x)) \overset{m-1}{\leq} k^n \]

\[ = (x, f(x)) \overset{\infty}{\leq} k^n \]

\[ \leq d(x, f(x)) \overset{L=\infty}{\leq} k^n \]

\[ = d(x, f(x)) \overset{k^n}{\leq} \frac{k^n}{1-k} \]

Given $\varepsilon > 0$, pick $N$ so that

\[ n \geq N \implies d(x, f(x)) \frac{k^n}{1-k} < \varepsilon. \]

So for $m, n \geq N$ we have

\[ d(f^m(x), f^n(x)) < \varepsilon. \]
Since \((x_0, a)\) is compact, \(\exists x_0 \to x_0\).

Since \(f^n(x)\) is continuous with \(f_0 \circ f_1 \circ \ldots \circ f_n(x) = f^{n+1}(x)\), for \(n \geq 0\).

But \(\exists f^{n+1}(x) \to x_0\).

So \(\lim x = x_0\) by uniqueness of limits.
(2) Say \( f(x_0) = x_0 \) \( f(x_0') = x_0' \).

\[
d(x_0, x_0') = d(f(x_0), f(x_0')) \\
\leq k \cdot d(x_0, x_1)
\]

Contradiction to \( k < 1 \)

(3) Since \( x \), \( f^n(x) \) converges to a fixed point and there is just one of new \( (3) \) follows.
Example: \( f: \mathbb{R} \to \mathbb{R} \), continuously differentiable and \( f \) is such that:

\[ |f'(x)| \leq K < 1 \quad \forall x \in \mathbb{R} \]

\[ \Rightarrow \text{unique fixed point.} \]

Proof: Mean Value Theorem.

\[ \forall x, y \quad x < y \quad \exists c \in (x, y) \]

\[ |f(y) - f(x)| = |f'(c)| |y - x| \]

\[ |f(y) - f(x)| = |f'(c)| (y - x) \leq K |y - x| \]

or \( d(f(y), f(x)) \leq K d(y, x) \).
If the graph of $f$ hits $y = x$, then $f(x_0) = x_0$.

Example $f(x) = x + 2$, no fixed points.
Another important property of complete metric spaces

Baire Category Theorem

No relation to "Category Theory"
No relation to "Bears"
DEF

• $A \subseteq X$ is nowhere dense if $\overline{\text{Int}(A)} = \emptyset$.

• $X$ is a Baire space if:
  When $\mathcal{A}$ is a collection of nowhere dense closed sets $\Rightarrow \bigcup \mathcal{A} = X$ has empty interior.

Example: In a Baire space $(T_1)$, countable sets have empty interiors.
We will prove $TR^n$ is Baire.

So a countable set in $IR^n$ has empty interior.

Another formulation which is equivalent whenever $E$ is Baire and collection $\sum_{n=1}^{\infty} \cup_{S_n}$ is dense.

Lemma: $E$ is a countable collection of open dense sets.
Example: $X = \mathbb{R}$ (a Baire space)

Let $\{ r_1, \ldots, 3 \}$ be an enumeration of $\mathbb{Q}$. Let $U_n = \mathbb{R} - \{ r_1, \ldots, r_n, 3 \}$.

It is open dense $\Rightarrow$

Irrationals $\mathbb{Q} = \bigcap_{n \in \mathbb{Z}_+} U_n$ is dense.

**Note:** The theorem often uses the language of first category and second category sets which we won't use.
Baire Category Theorem. If $X$ is complete metric or compact Hausdorff

$\Rightarrow X$ is Baire.

Proof needs a bit of machinery from later in the course.

**DEF.** $(X, T)$ is regular if

1. $T_1$ (points are closed)
2. $\forall$ points $x \in X$, all closed sets $B$

$\exists U, V$ open $U \cap V = \emptyset$

$x \in U \hspace{1cm} B \subseteq V$
Exampes

(1) \text{cat HD} \Rightarrow \text{regular}
(2) \text{metric} \Rightarrow \text{regular}

\centerline{Int(A) \neq A}

\quad d(x, B) > 0 \quad \text{since } x \notin \overline{B} = B.