Baire spaces

DEF: X is a Baire space if the following holds: Whenever countable $\{A_n\}$ of closed sets with empty interior $\Rightarrow$ $\bigcup_{n=1}^{\infty} A_n$ has empty interior.

Example We show $\mathbb{R}$ is Baire $\Rightarrow$ any countable set has empty interior.

ALT DEF: Whenever countable $\{U_n\}$ of open, dense $\Rightarrow$ $\bigcap_{n=1}^{\infty} U_n$ is dense.

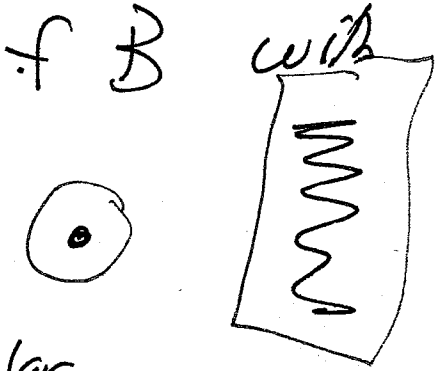
Baire Category Theorem

(1) X cpt, HD $\Rightarrow$ Baire

(2) X complete, metric $\Rightarrow$ Baire

Prelim Facts on Regularity

<u>DEF:</u> $(X, \mathcal{T})$ is regular if $\forall$ pts x and closed B $x \notin B$ $\Rightarrow \exists$ nbds U of x and V of B with $U \cap V = \emptyset$



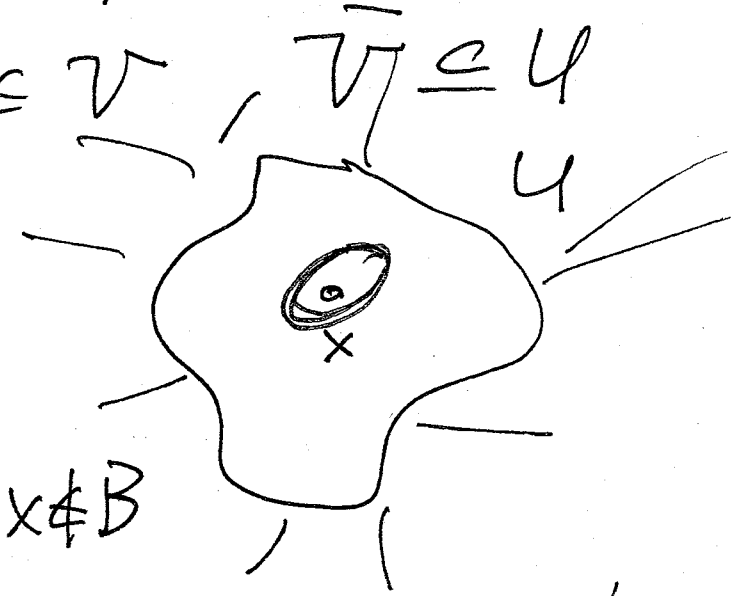
<u>FACTS:</u> $\rightarrow$ Compact HD $\Rightarrow$ Regular

metric $\Rightarrow$ Regular

Lemma X is regular $\iff$

given x and a nbd U of x

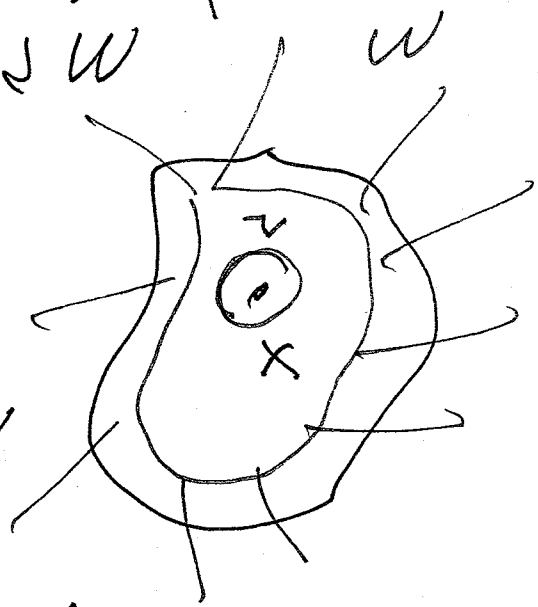
$\Rightarrow \exists$ open V $\quad x \in V$, $\bar{V} \subseteq U$



($\Rightarrow$) And $x \in U$, open

Let $B = X - U$ closed, $x \notin B$

$\Rightarrow \exists$ disjoint nbd V and W of x and B



Now $V \subseteq W^c$ a closed set

$\Rightarrow \bar{V} \subseteq W^c$ but $B \subseteq W$

$\Rightarrow \bar{V} \cap B = \emptyset \Rightarrow \bar{V} \subseteq U$

($\Leftarrow$) Given pt x and closed set B with $x \notin B$, $U = X - B$

~~then~~ then $x \in U \Rightarrow$ by hypothesis $x \in V \subseteq U$ and

$\bar{V} \subseteq U \Rightarrow$

V and $\bar{X} - \bar{V}$ are the needed open set $\blacksquare$

$\boxed{\text{Proof of BCT}}$ Start with $\{A_n\}$ closed, $\mathrm{Int}(A_n) = \emptyset$. We show that given any nonempty open U_0, $\exists x \in U_0$ with $x \notin \bigcup A_n$ (x is not in any of the A_n) $\Rightarrow \mathrm{Int}(\bigcup A_n) = \emptyset$.

By hypothes. A_1 doesn't contain U_0

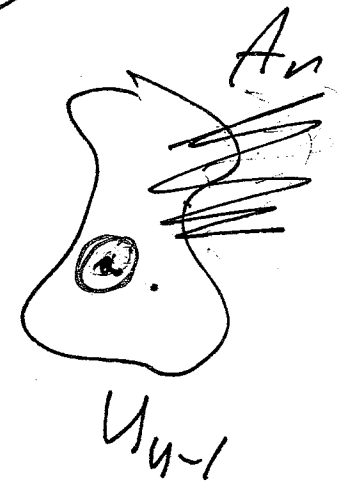
$\Rightarrow \exists y \in U_0 - A_1$ open

use Reg Lemma $\exists$ nbd U_1 of y $\overline{U_1} \cap A_1 = \emptyset$

$\overline{U_1} \subseteq U_0$

At the nth step, Given U_{n-1} we find a point in U_{n-1} not in A_n and an open set U_n which contains this point and $\overline{U_n} \cap A_n = \emptyset$

$$\overline{U_n} \subseteq U_{n-1}$$



[In the metric case, choose U_n ~~choose~~ so that $\operatorname{diam}(U_n) < 1/n$]

<u>Claim</u> $\bigcap_{n=1}^{\infty} \overline{U_n} \neq \emptyset$. <u>This will finish it</u>

$x \in \bigcap \overline{U_n} \Rightarrow x \in U_0$

and $\forall n$, $x \notin A_n$ since $U_n \cap A_n = \emptyset$

Proof of the claim

(1) Compact HD: $\overline{U_n}$ are a nested family of closed (so compact), nonempty so by finite intersection result

$$\bigcap_{n=1}^{\infty} \overline{U_n} \neq \emptyset \qquad \boxed{x = 3}$$

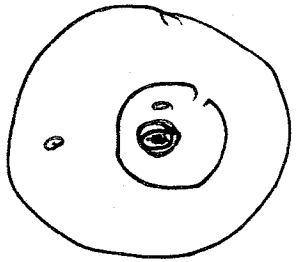
$a, b, c \leftarrow$

$x, y, z \leftarrow$

(2) Complete metric case

Lemma $C_1 \supset C_2 \supset \ldots$ nested nonempty in a complete metric space and $\operatorname{diam}(C_n) \to 0 \Rightarrow \bigcap_{n=1}^{\infty} C_n \neq \emptyset$

Proof Pick $x_n \in C_n$



$\Rightarrow$ Since $\operatorname{diam}(C_n) \to 0$

$\Rightarrow$ Retail of $\{x_n\}$ is in successive balls ~~To get~~ and Given $\varepsilon > 0$ pick N so that $n \geq N \Rightarrow \operatorname{diam}(C_n) < \varepsilon$

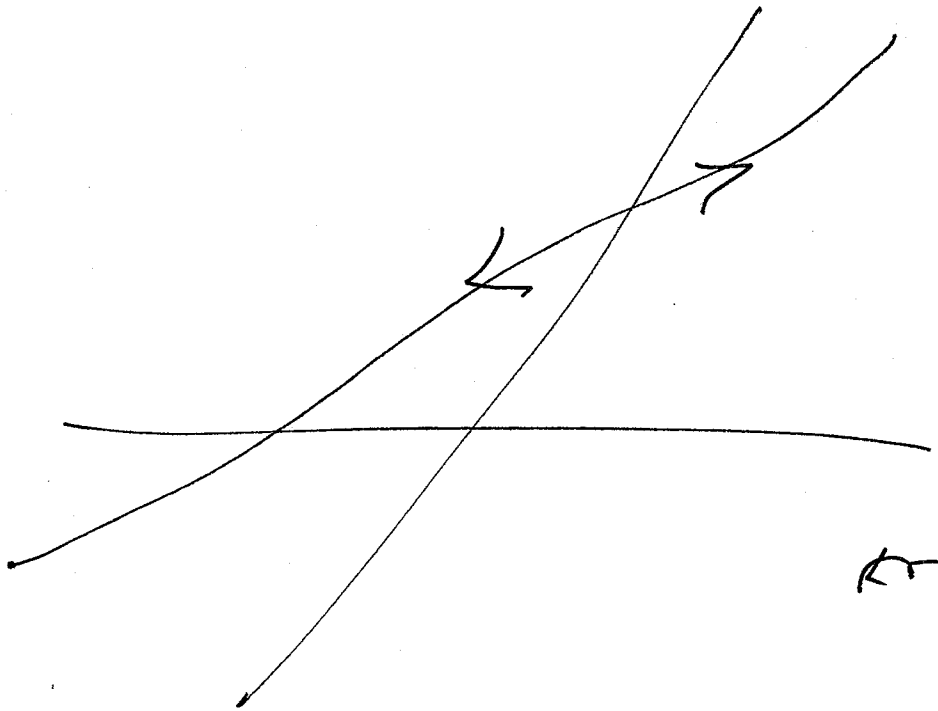
$\Rightarrow \{x_N, x_{N+1}, \ldots\} \subseteq C_N$

So $d(n,m) < \varepsilon$ if $n, m > N$.

So $\{x_n\}$ is Cauchy. So $\exists\, x$ with $x_n \to x$. Since $x_k, x_{k+1}, \ldots \to x$ also

So $x \in C_k$. True for all k

So $x \in \bigcap_{n=1}^{\infty} C_n$. $\blacksquare$



$f' > 1$

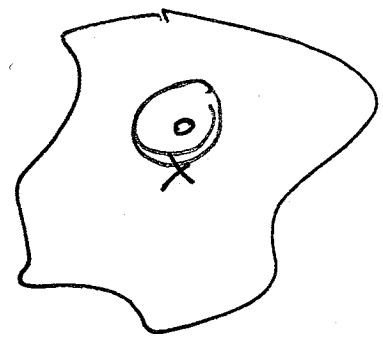
$f: \mathbb{R} \to \mathbb{R}$

$\Rightarrow \exists!$ fixed pt

Countability and separation axioms or properties

DEF $(X, \mathcal{T})$ is first countable at x if $\exists$ a countable collection of open elements $\mathcal{B}$ (the local base) so that for any nbd U of x $\exists$ $B \in \mathcal{B}$ with $x \in B \subseteq U$

Example (X, d) is metric

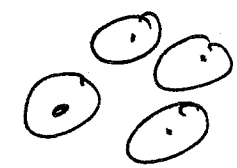


$$\mathcal{B} = \{ B_r(x) : r \in (0, \infty) \cap \mathbb{Q} \}$$

DEF $(X, \mathcal{T})$ is first countable if 1st countable at every point.

2nd **DEF**: $(X, \mathcal{T})$ is 2nd countable if it has a countable base.

Example $\boxed{\mathbb{R}^{\mathbb{Z}_+}}$ with unif top is 1st countable since its metric but not 2nd countable.



Theorem: $(X, \mathcal{T})$ is first countable.

(a) $x \in \bar{A} \iff \exists \{x_n\} \subseteq A \quad x_n \to x$

(b) f is cont $\iff (\forall\, x_n \to x,\ f(x_n) \to f(x))$

Proof like metric case.