DEF: $X$ is a Baire space if for all $\alpha \in \mathbb{N}$, there is a countable set $S$ of closed sets with empty interior such that $\bigcup_{n=1}^{\alpha} S_n$ has empty interior.

Example: We show $\mathbb{R}$ is Baire.

Alternatively, $X$ is Baire if for every countable set $S$ of open and dense subsets of $X$, there is a countable subset $\bigcup_{n=1}^{\alpha} S_n$ of $X$ such that $\bigcup_{n=1}^{\alpha} S_n$ is dense.
Baire Category Theorem

1. $X$ compact Hausdorff $\Rightarrow$ Baire
2. $X$ complete metric $\Rightarrow$ Baire

Prelim Facts on Regularity

**Def:** $(X, \tau)$ is regular if for all points $x$ and closed sets $B$ in $X$:

- There exists neighborhoods $U$ of $x$ and $V$ of $B$ with $U \cap V = \emptyset$.

**Facts:**
- Compact Hausdorff $\Rightarrow$ Regular
- Metric $\Rightarrow$ Regular
Lemma: \( X \) is regular \( \iff \)

given \( x \) and a nbd \( U \) of \( x \)
\[ \Rightarrow \exists \text{ open } V \subseteq U, \overline{\overline{V}} \subseteq U \]

\[ \Rightarrow \exists \text{ disjoint } T \text{ and } W \]

Let \( B = X - U \text{ closed }, x \not\in B \)

Now \( V \subseteq W \subseteq \text{ closed set } \)
\[ \Rightarrow \overline{V} \subseteq W \subseteq B^c \]
\[ \Rightarrow \overline{V} \cap B = \emptyset \Rightarrow \overline{V} \subseteq U \]
Given \( A \times X \) and closed set \( B \)

with \( A \times B \)

then \( x \in A \times L \) by \( x \in X \)

hypotheses \( x \in A \times L \) are the needed

if \( x \in X \times L \) open set

\( \subseteq L \)
Proof of BCT

Start with \( \exists A_n \)

Closed, \( \text{Int}(A_n) = \emptyset \). We show that

given any nonempty open \( U_0 \), \( \exists x \in U_0 \)

with \( x \notin V \bigcup A_n \) (\( x \) is not in any of

the \( A_n \)) \( \Rightarrow \text{Int}(V \bigcup A_n) = \emptyset \).

By hypothesis, \( A_1 \) does not contain \( U_0 \)

\( \Rightarrow \exists y \in U_0 - A_1 \) is open

Use regular lemma. \( \exists U_1 \) and \( \exists U_1 \)

of \( y \in \overline{U_1} \cap A_1 \) = \emptyset

\( U_1 \subseteq U_0 \)
\[ \text{Claim: } \forall n \in \mathbb{N}, \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } x < y < x + 1 \]

In the next case, choose \( y_n \) as follows:

\[ y_n = \frac{1}{n} \]

This point and \( y_n \) are not in \( A^n \) and are on one set which cannot find a point in \( L_n-1 \) not in An.
Proof of the claim

(1) Compact $\mathcal{H}$: $\{U_n\}$ are a nested family of closed (so compact), nonempty sets. So by finite intersection, result

$$\bigcap_{n=1}^{\infty} U_n \neq \emptyset$$

(2) Complete metric case

Lemma $C_1 \supset C_2 \supset \ldots$ nested, nonempty in a complete metric space

and $\operatorname{diam}(C_n) \to 0 \Rightarrow \bigcap_{n=1}^{\infty} C_n \neq \emptyset$
Proof: Pick $x_n \in C_n$

$\Rightarrow$ Since diam $(C_n) \rightarrow 0$

$\Rightarrow$ The tail of $\exists x_n \exists$ is in successive balls $\forall \varepsilon > 0$ and given $\varepsilon > 0$

Pick $N$ so that $n \geq N \Rightarrow \text{diam} (C_n) < \varepsilon$

$\Rightarrow \exists x_n, x_{n+1}, \ldots \exists \subseteq C_N$

$\Rightarrow d(n, m) < \varepsilon \text{ if } n, m \geq N.$

So $\exists x_n \exists$ is Cauchy. So $\exists x \text{ with } x_n \rightarrow x.$ Since $x_n, x_{n+1}, \ldots \rightarrow x \text{ also}$
So $x \in \bigcap_{k=1}^{\infty} C_k$. Thus for all $k$

So $x \in \bigcap_{n=1}^{\infty} C_n$.

$f' > 1$

$f: \mathbb{R} \to \mathbb{R}$

$\Rightarrow f_1$, fixed or

$f(x) \to x$
DEF \((X, \mathcal{O})\) is first countable at open countable collection of elements \(X \in \mathcal{O}\)

\[ B \text{ (the local base) } \text{ so that for any } \text{ and any } \exists U \text{ with } x \in U \subseteq Y \]

Example \((X, \mathcal{O})\) is not true

\[ B = \{ B_r(x) : r \in \mathbb{Q}, 0 < r < 1 \} \]

DEF \((X, \mathcal{O})\) is first countable at every point.
DEF: \((X, Y)\) is 2nd countable if it has a countable base.

Example: \(\mathbb{R}^2\) with unit topology is 1st countable since its metric is not 2nd countable.

Theorem: \((X, Y)\) is first countable.

(a) \(x \in A \iff \exists k_n \in A \ x_n \to x\)

(b) \(f\) is cont. \(\iff (\forall x_n \to x, f(x_n) \to f(x))\)

Proof: like metric case.